Aggregated Beliefs and Informational Cascades

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Extended version of the LORI-IV paper of same name.
The present version is incomplete: a notational change was made for the poster.
The proofs lack correction.

Abstract. A model of *informational cascades* is suggested with belief revision steps explicated using *dynamic epistemic logic* augmented with *transition rules* which specify the next action model update based on the current state. A notion of *aggregated beliefs* is used for information accumulation, based on which agents make choices and reason about others' belief construction. The informal reasoning assumed of agents in textbook treatments of information cascades is hereby represented formally. A self-propagating *system* is defined using transition rules, and it is shown that it does not terminate until the last agent has acted. Necessary and sufficient conditions (on the string of private signals) for informational cascades relative to the system are identified.

Keywords: informational cascades, social dynamics, social proof, dynamic epistemic logic

In the 1992 paper [4] Bikchandani et al. show how it may be rational for Bayesian agents in a sequential decision making scenario to ignore their private information and conform to the choices made by previous agents. If this occurs, an agent ignoring her private information is said to be *in a cascade*.

To illustrate, consider the following example, based on [3,8]: a set of agents must decide which of two restaurants to choose, one lying on the left side of the street, one on the right, with one being the better – L or R. Initially, agents have no information about which; each agent i has prior probabilities $Pr_i(L) = Pr_i(R)$. Every agent has two choices: either to go to the restaurant on the left, l_i, or to go the one on the right, r_i. All agents prefer to go to the better restaurant, and are punished for making the wrong choice, specified by pay-offs $u_i(l_i, L) = u_i(r_i, R) = v_1 > 0$ and $u_i(l_i, R) = u_i(r_i, L) = v_2 < 0$ with $v_1 + v_2 = 0$.

Before choosing, every agent receives a *private signal* indicating that either the restaurant on the left (L_i) or the one on the right (R_i) is the better one. The signals

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are assumed to be equally informative and positively correlated with the true state, in the sense that $Pr(L_i|L) = Pr(R_i|R) = q > .5$ and $Pr(L_i|R) = Pr(R_i|L) = 1 - q$. Given this setup, rational agents will follow their private signal, the majority choosing the better restaurant.

If agents are assumed to *choose sequentially* and *observe the choice of those choosing before them*, a cascade may result, possibly leading the majority to pick the worse option. The argument for this [4] rests on higher-ordering reasoning *not represented in the Bayesian framework*, and goes as follows. Given either L_1 or R_1, agent 1 will choose as her signal indicates, hereby revealing her signal to all subsequent agents. Agent 2 therefore as two pieces of information: his own signal together with that deduced from the choice of 1. If 2 receives the same signal as 1, he will make the same choice; given two opposing signal, assume he will invoke a self-biased tie-breaking rule, and go by his own signal. In both cases, 2's choice will also reveal his private signal to all subsequent agents. Assume that 1 and 2 received signals L_1, L_2. Then *no matter which signal* 3 *receives, she will choose* l_3: agent 3 will have three pieces of information, either L_1, L_2, L_3 or L_1, L_2, R_3. In either case, when conditionalizing on these, the posterior probability of L being the true state will be higher than that of R. So 3 will choose l_3, and thereby be in a cascade. Further, agent 4 will also be in a cascade: as 3 chooses l_3 no matter what, *her choice does not reveal her private signal*, why also 4 has three pieces of information, either L_1, L_2, L_4 or L_1, L_2, R_4. 4 is thus in the same epistemic situation as 3, and will choose l_4. As 4 is in a cascade, his choice will not reveal his private signal, *and the situation thus repeats for all subsequent agents*.

Notice that *cascades may not be truth conducive*: there is a $Pr(L_1|R) \cdot Pr(L_2|R)$ risk that *all* agents will choose the wrong restaurant – e.g., if signals are correct with probability $\frac{2}{3}$, all agents choose wrong with probability $\frac{1}{9}$.

Aim and Methodology. We construct a formal model that completely represents the reasoning made by agents in the sequential setup, for any input string of private signals. The type of model constructed may be compared to a dynamic epistemic logic variant of a state machine, in lack of terms called a *DEL machine*. A DEL machine operates by having for each state (Kripke model) some set of *transition rules* which as a function of the current state pick the next update to be invoked, hereby specifying the ensuing state. It is initiated from some *initial state* and terminates when an *end condition* is met.

The informational cascades system ($\mathbb{IC}$) constructed captures the following four elements of each agent's turn: *i)* earlier agents' actions are observed from which *ii)* their private signals (beliefs) are deduced and combined with *iii)* the private signal (belief) of the current agent after which *iv)* the chosen action is executed, observed by all.

$\mathbb{IC}$ diverges from the model of [4] in a number of aspects: it is *not probabilistic*, but *qualitative*; related, information aggregation is not done by Bayesian conditionalization, but by the *aggregation of perceived beliefs of others* (these beliefs reflect the received private signals); a *finite* set of agents is used; and *no pay-off structure nor rationality is assumed*, agency instead captured by transition rules.

An advantage of $\mathbb{IC}$ over the model from [4] is that $\mathbb{IC}$ fully specifies the intended scenario formally: all steps are defined for any string of private signals and all higher-order reasoning is represented. $\mathbb{IC}$ is thus a completely formal model of informational cascades.

The paper progresses as follows. In Sec. 1, elements from Dynamic Epistemic Logic (DEL) are introduced, and two first steps of agent 1's turn are modeled: the initial state of uncertainty and update with a private signal. In Sec. 2, it is shown how the agent may make a "modeler-independent" choice when DEL is augmented with *transition rules*, used to define two *agent types*. In Sec. 3, it is shown how agent 2 may extract information from the action of 1 by being informed of his agent type. In Sec. 4, agent 2 finishes his turn, and it is shown that agent 3 is in a cascade if she acts on *aggregated beliefs*. It is further shown that her action is uninformative to agent 4, who is then also in a cascade. In Sec. 5, the introduced elements are used to define a DEL machine, about which it is shown that it does not terminate prematurely and for which necessary and sufficient conditions of cascades are identified.

1 Epistemic States and Update Transitions

Fix a finite set of *agents* $\mathscr{A}$ and a countable set of *atoms* Φ.

Epistemic Plausibility Models. A *plausibility frame* (PF) is a structure $\langle S, \{\leq_i\}_{i \in \mathscr{A}} \rangle$ where S is a set of *worlds* with typical elements s, t, and $\leq_i$ is a *well-preorder*[1] on S for each *agent* $i \in \mathscr{A}$. An *epistemic plausibility model* (EPM) is PF augmented with a *valuation* $\|\cdot\| : \Phi \longrightarrow 2^S$ assigning to every atom in Φ a set of states from S. For

[1] A binary, reflexive and transitive relation in which every non-empty subset has a non-empty set of minimal elements. See [1] for more or [2] for a definition using *weakly connectedness*.

an EPM $\mathbf{S} = \langle S, \{\leq_i\}_{i\in\mathscr{A}}, \|\Phi\|\rangle$, let $D(\mathbf{S}) = S$, called the *domain* or *state space* of $\mathbf{S}$. A *pointed EPM* is a pair $(\mathbf{S}, s)$ with $s \in D(\mathbf{S})$.

Given an EPM, the *indistinguishability relation* for agent i is the equivalence relation obtained from the symmetric and transitive closure of $\leq_i$, denoted $\sim_i$. The *information cell* of agent i at state s is

$$\mathscr{K}_i[s] = \{t : s \sim_i t\}$$

and the *plausibility cell* of agent i at state s is

$$\mathscr{B}_i[s] = Min_{\leq_i}\mathscr{K}_i[s] = \{t \in \mathscr{K}_i[s] : t \leq_i s', \text{ for all } s' \in \mathscr{K}_i[s]\}.$$

The plausibility cell $\mathscr{B}_i[s]$ contains the worlds the agent find *most plausible* from the information cell $\mathscr{K}_i[s]$ and represent the "doxastic appearance" [1, p. 25] of s to i.[2] Notice that $s \leq_i t$ means that s is at *least as plausible* as t for i.

Language and Satisfaction. Where $P \in \Phi$ and $i \in \mathscr{A}$, let the well-formed formulas of language $\mathscr{L}_{BK}(\Phi, \mathscr{A})$ be given by the grammar

$$\varphi := \top \mid \bot \mid P \mid \neg\varphi \mid \varphi \wedge \psi \mid B_i\varphi \mid K_i\varphi.$$

The satisfaction relation $\models$ between pointed EPMs and $\mathscr{L}_{BK}$ is defined mostly as usual (see e.g. [5] for details), why only the cases for doxastic and epistemic formulas are presented:

$$(\mathbf{S}, s) \models B_i\varphi \text{ iff } \forall s' \in \mathscr{B}_i[s] : (\mathbf{S}, s') \models \varphi$$

$$(\mathbf{S}, s) \models K_i\varphi \text{ iff } \forall s' \in \mathscr{K}_i[s] : (\mathbf{S}, s') \models \varphi$$

Boolean connectives are defined as usual. For $(\mathbf{S}, s) \models \varphi$, say that P is *true* or *satisfied* at state s in model $\mathbf{S}$. Parentheses and reference to $\mathbf{S}$ may be omitted when clear from context. *Entailment* is given by $\varphi \models \psi$ iff $(\mathbf{S}, s) \models \varphi$ implies $(\mathbf{S}, s) \models \psi$ for all $(\mathbf{S}, s)$. Denote by $\|\varphi\|_{\mathbf{S}}$ the set of states from $D(\mathbf{S})$ that satisfy φ, i.e. $\|\varphi\|_{\mathbf{S}} = \{t \in D(\mathbf{S}) : (\mathbf{S}, t) \models \varphi\}$.

Agents and Atoms. To construct a model of informational cascades, fix and enumerate a set of agents $\mathscr{A} = \{1, 2, ..., n\}$ with $n \geq 3$. It will be assumed that agents

[2] The definition of EPMs is based on [1]. The notation for information and plausibility cells are adopted from [7].

make their choice in accordance with enumeration, i.e. 1 chooses first, followed by 2, etc.

To denote which of the two options available options are *in fact* the correct one, use the atom L (for 'the restaurant on the left is the better one') and it's negation $\neg L =: R$ (for 'the restaurant on the right is the better one').

To represent which of the two possible options was chosen by agent i, the atoms $\alpha_i L$ and $\alpha_i R$ are used, read respectively 'i chose the restaurant on the left' and 'i chose the restaurant on the right'. These literals are *post-factual action descriptions*, not the actions themselves. The executed actions are instead captured by *action models*, as specified below. As i may not yet have made any choice, it is natural that $\neg\alpha_i L \wedge \neg\alpha_i R$ should be satisfiable, why $\alpha_i R$ is not defined as shorthand for $\neg\alpha_i L$ (nor *vice versa*). As any agent may at most make one choice, impose the restriction that $\left\|\alpha_i L\right\|_{\mathbf{S}} \cap \left\|\alpha_i R\right\|_{\mathbf{S}} = \emptyset$ for all $\mathbf{S}$. In the remaining, let the set of atomic propositions be given by $\Phi = \{L\} \cup \{\alpha_i L, \alpha_i R\}_{i \in \mathscr{A}}$.

Turns and Initial Uncertainty. Preemptively, each agent's *turn* will comprise four steps/models: *i*) $\mathbf{S_i}$, the initial state of i's turn, *ii*) $\mathbf{I_{i-1}}$ invoking the *interpretation* of agent $i-1$'s executed action,[3] *iii*) $\mathbf{P_i}$, the *private signal* of i, forming her private beliefs about L/R, and *iv*) either l_i or r_i, the action i finally executes. After the execution, the initial state of $i+1$ results. The initial state for agent 1 is depicted in Fig. 1.

Fig. 1. The EPM $\mathbf{S_1}$ representing the initial uncertainty about the better restaurant. All agents know it's either, but does neither know nor believe which. Labels L and R indicate truth of the atom, e.g. $s_0 \in (L)_{\mathbf{S_1}}$. For all $P \in \{\alpha_i L, \alpha_i R\}_{i\in\mathscr{A}}$, $\|P\|_{\mathbf{S_1}} = \emptyset$ as no agent has chosen.

Action Plausibility Models. An *action plausibility model* (APM) is a plausibility frame $\langle \Sigma, \{\preceq_i\}_{i\in\mathscr{A}}\rangle$ with Σ finite, augmented by a *precondition map* $pre : \Sigma \longrightarrow \mathscr{L}_{BK}(\mathscr{A}, \Phi)$ and a *postcondition map* $post : \Sigma \longrightarrow \mathscr{L}_{BK}(\mathscr{A}, \Phi)$ such that $post(\sigma) = \psi$ where $\psi \in \{\top, \bot\}$ or $\psi = \bigwedge_1^n \varphi_n$ with $\varphi_i \in \{P, \neg P : P \in \Phi\}$.[4] An APM is thus a structure $\mathbf{E} = \langle \Sigma, \{\preceq_i\}_{i\in\mathscr{A}}, pre, post\rangle$. A *pointed APM* is a pair $(\mathbf{E}, \sigma)$ with $\sigma \in D(\mathbf{E}) = \Sigma$.

[3] Which for obvious reasons is skipped for agent 1.
[4] Again, this presentation follows [**?**], with the addition of postconditions as formulated in [14,6].

Just as every world in an EPM represents a possible state of affairs, specified by the world's true propositions, so every action in an APM represents a possible *change*. *What* change is specified by the pre- and postconditions; preconditions determine what is required for the given action to take place, i.e. what conditions a world must satisfy for an action to executable in that world, and postconditions what *ontic* change the action brings about.[5] The effect of an APM on a given EPM is found by taking the *action-priority update product*, specified below.

The event where 1 receives (correct) private signal L is captured by the APM in Fig. 2 with $i := 1$, setting the actual state to σ_1; the (incorrect) private signal that R is captured by setting the actual state to σ_2.[6]

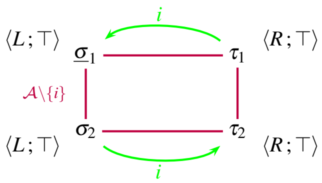


Fig. 2. APM $\mathbf{P_i}$: i receives private signal while others remain uninformed about *which*. State labels $\langle \varphi ; \psi \rangle$ specify pre- and postconditions. Transitive and reflexive arrows are not drawn.

Expl.: Agent $i := 1$ in fact receives signal/soft information that L and is certain about this. All others find it equally plausible that 1 is informed of L or R and that 1 is certain about which. No *ontic* change occurs.

Fig. 2 does not display the full relations for all agents. Most notably, reflexive loops and and links obtained by transitive closure are omitted. A rule-bound method for depiction is not implemented, instead 'easy-to-read' figures are sought produced.

Doxastic Programs. $\mathbf{P_1}$ includes uncertainty for all agents, but this may be restricted by looking at *doxastic programs* over the model. A doxastic program is the action model equivalent of a proposition, i.e. a subset of all actions in the models' event space: $\Gamma \subseteq \Sigma$. Singleton programs are denoted by the state name, and the full state space program by the model name. Over $\mathbf{P_1}$, the program $\Gamma_1 = \{\sigma_1, \tau_1\}$ captures the event where 1 *publicly* receives signal that L, while the program $\mathbf{P_1}$ (the full state space) captures the event where the signal is received *privately*. In

[5] To exemplify, the action 'agent a plays a Queen' is only executable when a has a Queen on hand (precondition), and brings the factual change that the given Queen has now been played (postcondition). An ontic change is non-doxastic change, here a change in atomic truth value.

[6] Specifying private signals thusly breaks with an aspect of the model presented in [4], as they assume common knowledge among agents that the signal received is positively correlated with the truth. To save space, this aspect is ignored, though it may be modeled using extra atoms $\{S_iL, S_iR\}_{i\in\mathscr{A}}$ for private signals and making $\langle L; S_iL\rangle$ more plausible than $\langle L; S_iR\rangle$, etc.

the ensuing, it will be assumed that doxastic programs have a designated state. To incorporate new information from a doxastic program into $\mathbf{S_1}$, the *action-priority update product* is taken.

Action-Priority Update Product. The *action-priority update* is a binary operation $\otimes$ with first argument an EPM $\mathbf{S}$ and second argument a doxastic program $\Gamma \subseteq \Sigma$ with designated state σ_0, over some APM $\mathbf{E}$ with action space Σ. The *APU product* is an EPM $\mathbf{S} \otimes \Gamma = (S \otimes \Gamma, \leq_i^{\uparrow}, ||\Phi||^{\uparrow}, (s_0, \sigma_0))$ where the updated state space is $S \otimes \Gamma = \{(s,\sigma) \in S \times \Gamma : \mathbf{S}, s \models pre(\sigma)\}$; each updated pre-order $\leq_i^{\uparrow}$ is given by $(s,\sigma) \leq_i^{\uparrow} (t,\tau)$ iff either $\sigma \prec_i \tau$ and $s \sim_i t$, or else $\sigma \simeq_i \tau$ and $s \leq_i t$;[7] the valuation set $||\Phi||^{\uparrow}$ is given by the following: for every atom $P \in \Phi$, $P_{\mathbf{S}\otimes\Gamma} = (\{(s,\sigma) : s \in P_{\mathbf{S}}\} \backslash \{(s,\sigma) : post(\sigma) \models \neg P\}) \cup \{(s,\sigma) : post(e) \models P\}$ for states $(s,\sigma) \in S \otimes \Sigma$. Finally, (s_0, σ_0) is the new actual world.[8]

The APU product gives priority to new information encoded in Γ over the old beliefs from $\mathbf{S}$ by the 'anti-lexicographic' specification of $\leq_i^{\uparrow}$ that gives priority to the APM plausibility relation $\preceq_i$. This is stands in contrast to the *product update* often used in DEL settings (see e.g. [**?**,15,13,6]) where both relations are given equal priority.[9] The definition further clarifies the role of pre- and postconditions; if a world does not satisfy the preconditions of an action, then the given state-action pair does not survive the update, and if postconditions are specified, these override earlier ontic facts, else leave all as was.

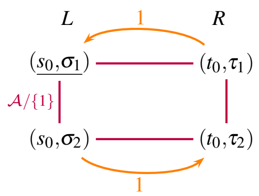


Fig. 3. The EPM $\mathbf{S_1} \otimes \mathbf{P_1}$ in which 1 has received her private signal. In the actual state (s_0, σ_1), $B_1 L$ is satisfied, along with $K_j(B_1 L \vee B_1 R)$, $\neg K_j B_1 L$ and $\neg K_j B_1 R$ for $j \in \mathscr{A} \backslash \{1\}$.

2 Informed Decisions

After having received her private signal, agent 1 stands to choose between the two restaurants. Her choice is done *publicly,* post-factually represented by an atom, $\alpha_1 L$

[7] $\preceq_i$ is from $\mathbf{E}$ and $\leq_i$ from $\mathbf{S}$. $\sigma \prec_i \tau$ denotes ($\sigma \preceq_i \tau$ and not $\sigma \succeq_i \tau$), $\sigma \simeq_i \tau$ denotes ($\sigma \preceq_i \tau$ and $\sigma \succeq_i \tau$).

[8] The definition is based on [1] for the APU product with the valuation clause from [14,6].

[9] Product update is often used in situations where both relations are assumed to be equivalence relations with the updated equivalence relation $\sim_i^{\uparrow}$ given by $(s,\sigma) \sim_i^{\uparrow} (t,\tau)$ iff $s \sim_i t$ and $\sigma \simeq_i \tau$. See [1] for comments on the relationship between the two.

or $\alpha_1 R$. *Ex post*, the action should hence be known to all. A suitable APM for both actions is depicted in Fig. 4.

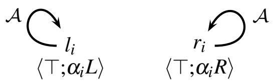


Fig. 4. The APM $\mathbf{A_i}$ over which the two possible actions for agent i is given; either i may choose to go left (l_i) or right (r_i)

Over $\mathbf{A_1}$, the program l_1 captures the public "announcement" that 1 is chose the left restaurant, and *vice versa* for r_1. Given her belief that L, for 1 to act reasonably, it is clear that the next transition in the sequence should be an update with l_1. However, simply performing this update *as modeler* does not present 1 with much of a *choice*. Put differently, if we as modelers have to inspect the model and hand pick a next update for each agent action, the agents are not very *autonomous*: their decision architecture is not incorporated in the sequence model, but only in the mind of the modeler. Modeler independent models may also be obtained by using *DEL protocols* (e.g. [13]) to specify next updates. Relations to the protocol approach are discussed below.

One way incorporate the decision architecture of agents suitable for epistemic logic is the *knowledge-based programs* of [9], being simple directions of the form '**if** $K_1 L$, **do** $\alpha_1 L$', specifying an action based on local epistemic state.[10] Defining such a rule for each relevant belief allows for the definition of various *agent types* with choices specified for also counter-factual situations. These rules may then be considered constituent parts of the sequence model, or *system*, specified below. To not conflate two notions of *programs*, the term *transition rules* will be used to denote the version here tailored to the DEL framework.

Transition Rules. A *transition rule* $\mathcal{T}$ from language $\mathcal{L}$ is an expression

$$\varphi \rightsquigarrow [X]\psi$$

where $\varphi, \psi \in \mathcal{L}$. Call φ the *trigger* and ψ the *effect*. If EPM $(\mathbf{S}, s)$ satisfies the trigger of a transition rule $\mathcal{T}$, $\mathcal{T}$ is said to be *active* in $(\mathbf{S}, s)$ (else *inactive*).

Specified below, transition rules may be used to choose the next update based on local conditions of the current EPM. E.g., updates by the 'environment' may be

[10] Another possibility would be to introduce a game- or pay-off structure in parallel to the DEL framework or embed the entire dynamics modeled in a temporally extended game tree, whereby actions could be made 'rationally', based on utility maximization at end nodes. A drawback to this method is the large models required: every branch must be fully specified before decisions may follow. Using knowledge-based programs, a "localized" modeling procedure may be followed instead.

specified using atoms in the trigger: let r and w be atoms with resp. readings 'it rains' and 'the street is wet'. Then the transition rule $\mathscr{T} = r \rightsquigarrow [X]w$ reads '**if** it rains, **then** *the next update must be such that after it*, the street is wet'.

Transition rules may also be used as *agent decision rules* invoking e.g. ontic change, by using $B_i\varphi/K_i\varphi$-formulas as triggers and suitable formulas as effects. E.g., the set of transition rules $\{B_i r \rightsquigarrow [X]u_i, B_i\neg r \rightsquigarrow [X]\neg u_i\}$ may be used to specify agent behavior relative to rain: if i believes it rains, then next i will have an umbrella, and if i believes it does not rain, then next i will not have an umbrella.

Using transition rules, two relevant agent types may now be defined, see Table 1. The *individual* agent acts on private beliefs about L and R, whereas the *aggregator* acts on *aggregated* beliefs over group G, to be defined. For now, let 1 be individual.

Individual:	**Aggregator:**
$\mathscr{I}_L = B_i L \rightsquigarrow [X]\alpha_i L$	$\mathscr{A}_L = A_{i\vert G} L \rightsquigarrow [X]\alpha_i L$
$\mathscr{I}_R = B_i R \rightsquigarrow [X]\alpha_i R$	$\mathscr{A}_R = A_{i\vert G} R \rightsquigarrow [X]\alpha_i R$

Table 1. Decision rules specifying two agent types: the individual, who acts on private beliefs only, and the aggregator, who bases decisions on *aggregated beliefs* (defined below).

A similar notion of agent types was introduced in the recent work [11] where agent types are included in the formal language. The primary difference between the two is that the agent types of [11] are defined by providing *necessary* conditions for action. In contrast, the present approach lists *sufficient* conditions. In the notation of [11], $\varphi \twoheadleftarrow_x !\varphi$ specify that x is of the 'truth teller' type; if x announces φ, then φ must be the case *before the announcement*. Contrary, the rule $\varphi \rightsquigarrow [X]\varphi$ expresses that if φ is true, then φ *must continue to be true after the next update*.

Dynamic Modalities. Note that *transition rules are not doxastic propositions*: the "modality" $[X]$ has no interpretation, and construed as a formula, transition rules have no truth conditions. Instead, transition rules are prescriptions *for choosing the next action model*. The choice of model is made by implementing a transition rule over an EPM **S** and a specified set of doxastic programs over one or more APMs using *dynamic modalities.*

For any program Γ over APM **E**, let $[\Gamma]$ be a dynamic modality. Semantics for $[\Gamma]\varphi$ are given by

$$\mathbf{S}, s \models [\Gamma]\varphi \text{ iff } \forall\sigma, \text{ if } (s,\sigma) \in S \otimes \Gamma, \text{ then } \mathbf{E}, (s,\sigma) \models \varphi.$$

That is, a state s from $\mathbf{S}$ is a $[\Gamma]\varphi$-state iff every resolution of Γ over s is a φ-world in $\mathbf{S} \otimes \Gamma$.

Solutions and Next APM Choice. A set of transition rules dictates the choice for the next APM by finding the transition rule(s)'s *solution*. A *solution* to $\mathscr{T} = \varphi \rightsquigarrow [X]\psi$ over pointed EPM $(\mathbf{S}, s)$ is a doxastic program Γ such that

$$\mathbf{S}, s \models \varphi \rightarrow [\Gamma]\psi.$$

Γ is a solution to the set $\mathbb{T} = \{\mathscr{T}_1, ..., \mathscr{T}_n\}$ with $\mathscr{T}_k = \varphi_k \rightsquigarrow [X]\psi_k$ over $(\mathbf{S}, s)$ if $\mathbf{S}, s \models \bigwedge_1^n(\varphi_k \rightarrow [\Gamma_k]\psi_k)$, i.e. if Γ is a solution to all $\mathscr{T}_i$ over $(\mathbf{S}, s)$ *simultaneously*.[11] Finally, a set of doxastic programs $\mathbb{S}$ is a solution to $\mathbb{T}$ over $\mathbf{S}$ iff for every t of $\mathbf{S}$, there is a $\Gamma \in \mathbb{S}$ such that Γ is a solution to $\mathbb{T}$ over $(\mathbf{S}, t)$.

If $\mathbb{S}$ is a solution to $\mathbb{T}$ over $\mathbf{S}$, then given a state s from $\mathbf{S}$, the transition rules in $\mathbb{T}$ specify one (or more) programs from $\mathbb{S}$ as the next choice: the set of solutions to $\mathbb{T}$ from $\mathbb{S}$ over $(\mathbf{S}, s)$. A *deterministic* choice will be made if $\mathbb{S}$ is selected suitably, i.e. if it contains a *unique* Γ for each s. In the ensuing, solution spaces will be chosen thusly.

Rule-Based Choice. Given that 1 is of the individual type, let us now see how transition rules facilitate choice. Three things are required; the current EPM, $\mathbf{S_1} \otimes \mathbf{P_1}$; a set of transition rules, $\mathbb{I} = \{\mathscr{I}_L, \mathscr{I}_R\}$, and a solution space, $\mathbb{S}_1 = \{l_1, r_1\}$. Is $\mathbb{S}_1$ a solution to $\mathbb{I}$ over $\mathbf{S_1}$? If so, then model, transition rules and solution space 'fit' as a next APM choice will be specified for each state in $\mathbf{S_1} \otimes \mathbf{P_1}$ where a transition rule is active, so also for the actual state in $\mathbf{S_1} \otimes \mathbf{P_1}$, denote it s_0, hereby providing the actual choice. Of course it is. All cases are analogous, so focus on the actual state. To see that $\mathbb{S}_1$ is a solution to $\mathbb{I}$ over $(\mathbf{S_1}, s_0)$, note that $s_0 \models B_1 L$. Hence *both* l_1 and r_1 (trivially) solve $\mathscr{I}_R$ over $(\mathbf{S_1}, s_0)$, as the rule is inactive (unsatisfied trigger and thus antecedent). As $\mathscr{I}_L$ is active, a solution Γ must satisfy $s_0 \models [\Gamma]\alpha_1 L$. Of l_1 and r_1, clearly only l_1 does so, why it is the next APM choice, see Fig. 5. So, as the modeler would have it, 1 chooses to go to the left restaurant. Note, though, that had 1 been given a *different* private signal (that R), then her beliefs would have changed, and so would her choice *without further tampering by the modeler*.

[11] Note the analogy with numerical equations; for both $2 + x = 5$ and $\{2 + x = 5, 4 + x = 7\}$, $x = 3$ is the (unique) solution.

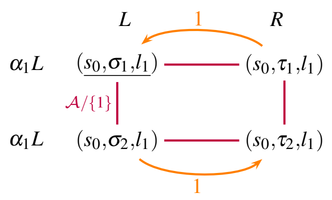


Fig. 5. The EPM $\mathbf{S_2} := (\mathbf{S_1} \otimes \mathbf{P_1}) \otimes l_1$, sharing the frame and valuation set of $\mathbf{S_1} \otimes \mathbf{P_1}$, only with $\alpha_1 L$ now satisfied at all states. Hence $\mathbf{S_2} \models \bigwedge_{i \in \mathscr{A}} K_i \alpha_1 L$ while still $\mathbf{S_2} \models \bigwedge_{i \in \mathscr{A}} \neg K_i B_1 L$.

3 Reconstructing Reasons: Action Interpretation

Being seated, 1 may now move on to choose from the menu, for which she is left on her own. In turn, agent 2 stands to choose restaurant, but before doing so, she will both receive a private signal, but also try to extract information from the choice of 1. For her to do so, she must *interpret* (or rationalize, or forward induce on) the *action* performed by 1: why would she choose restaurant L? Disregarding reasons such as forced hand or employee discount, an obvious candidate for an explanation is that 1 believes it is the better one. As no overarching structure allowing for e.g. forward induction is present in the DEL framework, an approach utilizing an 'inverse' version of decision rules, brute forcing conclusions about belief from observations about action, is suggested.

Interpretation Rules. By an *interpretation rule* is simply meant a doxastic proposition $\varphi \to B_i \psi$. The underlying idea is that on the basis of an action (e.g. $\varphi := \alpha_i L$), agents may deduce something about the content of i's beliefs (e.g. that $B_i L$).[12]

A set of interpretation rules may in general be implemented using an APM where the preconditions of each state is a conjunction of interpretation rules with different bases with a conjunct for each action to be interpreted. Hereby each state represents a different hypothesis regarding the acting agent's type, i.e. how the agent made decisions. The plausibility order then specifies the 'abductive hierarchy' of such hypotheses.

In the present, agents are given only *one* hypothesis about types, the hypothesis also *correct* in the sense that the interpretation rules are (close to) the converse of the in fact applied transition rules. Hence the interpretation rule APM $\mathbf{I_i}$ that determines how $\mathscr{A}$ interprets the actions of i is a one state model. Set the preconditions for this state, i_i, to

[12] More detailed interpretation rules are used in [12] which also respect the temporal aspect introduced by updated, whereby the *earlier* beliefs of the actor are concluded. This aspect is ignored in the present to simplify. This causes no formal problem, as the beliefs of agents who have already acted will stay fixed.

$$pre(i_i) := \alpha_i L \rightarrow B_i L \;\; \wedge$$
$$\alpha_i R \rightarrow B_i R$$

Updating $\mathbf{S_2}$ with $\mathbf{I_1}$ produces the model depicted in Fig. 6. The two lower states from $\mathbf{S_2}$ are deleted as they do not satisfy the consequent of the first conjunct of the preconditions. The second conjunct plays no role here, *but would have, had things been different*.

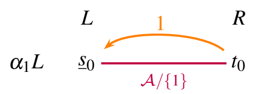


Fig. 6. The EPM $\mathbf{S_2}\otimes\mathbf{I_1}$: $\mathscr{A}$ has interpreted the action of 1 correctly, and now now her belief that L.

Hereon out, used state names will not reflect construction; the vector notation grow too long.

Why Include Interpretation? A criticism has been raised pertaining to the necessity of including interpretation rules, arguing that it is overly complex as the information obtained could have been included in a simpler way by using $B_i L$ and $B_i R$ as preconditions for the actions l_i and r_i of $\mathbf{A_i}$, resp. Indeed, the result would be the same, and the dynamics one step shorter.

However, physical action, as in restaurant choice, is not an announcement of belief; the action and the rationalization are separate, why they may be modeled as separate steps.[13] Though nothing in the ensuing hinges on using a distinct interpretation model, the separation provides a more fine grained view of the temporal structure of the dynamics in play, broken into ‘smallest pieces’.[14] Furthermore, as an argument for the approach in general, using a distinct interpretation model provides an easily modifiable module for specifying perception and higher-order perception of agent types.

4 Aggregated Beliefs and Cascading

Reusing the APM in Fig. 2 with $i := 2$ and actual state again σ_1, the model, call it $\mathbf{P_2}$, captures that 2 is privately informed that L, see Fig. 7. If it is assumed that also 2 is of the individual type and has the same possible moves available as 1 (Fig. 4,

[13] This even goes if the physical action is an announcement: in speech act theory at least, the performative ‘I choose L’ and the contingent perlocutionary consequence that the speaker’s beliefs become known are distinct.

[14] Ideas for further deconstruction of any steps in the dynamics suggested are *very* welcome.

$i := 2$), then given his belief that L in s_0 of $(\mathbf{S_2} \otimes \mathbf{I_1}) \otimes \mathbf{P_2}$, the next APM choice will be l_2, changing $(\mathbf{S_2} \otimes \mathbf{I_1}) \otimes \mathbf{P_2}$ only by setting $\alpha_2 L$ true everywhere.

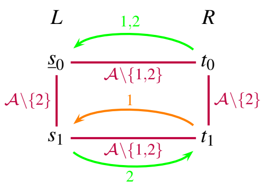


Fig. 7. The EPMs $(\mathbf{S_2} \otimes \mathbf{I_1}) \otimes \mathbf{P_2}$ and $((\mathbf{S_2} \otimes \mathbf{I_1}) \otimes \mathbf{P_2}) \otimes l_2$: Identical, though the latter with $\alpha_2 L$ true at all states. Still only 2 knows her beliefs.

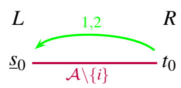


Fig. 8. The EPM $\mathbf{S_3} \otimes \mathbf{I_2}$: Following interpretation, everyone knows 2's beliefs: the states s_1 and t_1 in Fig. 7 does not satisfy $\alpha_2 L \rightarrow B_2 L$, so they did not survive.

Aggregated Beliefs. In the initial state for agent 3's turn, $\mathbf{S_3} := ((\mathbf{S_2} \otimes \mathbf{I_1}) \otimes \mathbf{P_2}) \otimes l_2$, the action of agent 2 may now be interpreted analogous to that of agent 1 by suitably indexing the one-state interpretation APM above, obtaining $\mathbf{I_2}$. The result, see Fig. 8, has the important feature from previously: all agents now *know* that 2 believes that L.

Assuming that 3 is individualistic with the same choices and is given a private signal that L, it should be clear that 3, too, will choose the left restaurant, and *ceteris paribus* with private signal that R (Fig. 2, $i := 3$ and actual state σ_2), that she would choose the right.

However, 3 has more information than her own private signal available. As she as correctly interpreted the actions of 1 and 2, she knows what beliefs their private signals caused. As neither has changed their beliefs since receiving their signals, and there is a one-to-one correspondence between believing L or R and the two private signals, 3 may accumulate information from all three signals by aggregating the beliefs of all three agents regarding L/R. Call an agent that acts in this manner an *aggregator*: one who bases decision on both her private information as well as information extracted by witnessing the actions of others.

To capture the required *aggregated beliefs* introduce a new operator $A_{i|G}$, representing the beliefs of agent i when aggregating information from her beliefs about the beliefs of agents from group G. The semantics for $A_{i|G}$ is defined using simple majority 'voting' with a self-bias tie-breaking rule. To simplify, let $A_{i|G}\varphi$ be well-formed only for atomic φ. For $p \in \Phi$,

$$\mathbf{S}, s \models A_{i|G} p \text{ iff } \alpha + |\{j \in G : \mathbf{S}, s \models B_i B_j p\}| > \beta + |\{j \in G : \mathbf{S}, s \models B_i B_j \neg p\}|$$

with tie-breaking parameters α, β given by

$$\alpha = \begin{cases} 1/2 & \text{if } s \in (B_i \varphi)_{\mathbf{S}} \\ 0 & \text{else} \end{cases}$$

$$\beta = \begin{cases} 1/2 & \text{if } s \in (B_i \neg\varphi)_{\mathbf{S}} \\ 0 & \text{else} \end{cases}$$

This definition leaves agent i's aggregated beliefs undetermined iff both i is agnostic whether p and there is no strict majority on the matter.

It was postulated that aggregated beliefs and Bayesian conditionalization are "equivalent" for the analysis of informational cascades. The case for this is that given the specific setup presented, it may be shown that Bayesian agents may "decide according to a majority-vote over the signals they receive [i.e. are able to deduce]" [8, p. 439]. Majority voting is sensible since every agents private information regarding L/R is equally good, and under the assumption that every agents information is correct with a probability above .5. Then using $A_{i|G}L/R$ to make a decision about which action to take is justified by Condorcet's jury theorem.

Being in Cascade. To make 3 act on aggregated beliefs, let 3 be an *aggregator*, as defined in Table 1, i.e. $\mathscr{A}_L = A_{i|.\mathscr{A}} L \rightsquigarrow [X]\alpha_i L$ and $\mathscr{A}_R = A_{i|.\mathscr{A}} R \rightsquigarrow [X]\alpha_i R$, with the same possible moves as 1 and 2 (Fig. 4, $i := 3$). Acting thusly puts agent 3 *in cascade*: *no matter what her private information*, she will act as those before her.

To see this, observe the EPM in Fig. 9, the product of $\mathbf{S_3} \otimes \mathbf{I_2}$ and the private signal model for 3, $\mathbf{P_3}$. For now, set $\mathbf{S} := (\mathbf{S_3} \otimes \mathbf{I_2}) \otimes \mathbf{P_3}$. Note first that given the private signal that L ($\mathbf{P_3}$ with actual state σ_1), s_0 would be actual, where for private signal that R ($\mathbf{P_3}$ with actual state σ_2), the actual state would be s_1.

Next, see that both satisfy $A_{3|.\mathscr{A}} L$: at s_0, $\alpha + |\{j : s \in (B_3 B_j L)_{\mathbf{S}}\}| = 3½$ as all three agents are believed to believe L, while $\beta + |\{j : s \in (B_3 B_j R)_{\mathbf{S}}\}| = 0$. Hence $s_0 \in \left\| A_{3|G} L \right\|_{\mathbf{S}}$. At s_1, $\alpha + |\{j : s \in (B_3 B_j L)_{\mathbf{S}}\}| = 2$ as agents 1 and 2 are believed to believe L, while $\beta + |\{j : s \in (B_3 B_j R)_{\mathbf{S}}\}| = 1½$ as 3 herself believes R. Hence also $s_1 \in \left\| A_{3|G} L \right\|_{\mathbf{S}}$. Given this, it is clear that for both pointed EPMs, the next APM choice will be l_3 as this is the only solution from $\mathbb{S}_3$ to $\mathscr{A}_L$. So, no matter her private signal, 3 chooses l_3.

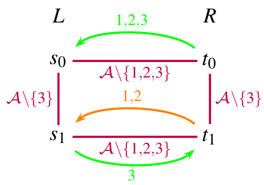


Fig. 9. The EPM $(\mathbf{S_3} \otimes \mathbf{I_2}) \otimes \mathbf{P_3}$ without specified actual state. If 3 is given private signal that L $(\mathbf{P_3}, \sigma_1)$, the actual state is s_0; for signal that R $(\mathbf{P_3}, \sigma_2)$, the actual is s_1.

Note that though 3 is in cascade if she aggregates, this does not hold for neither 1 or 2; had either been assumed aggegators, nothing in the dynamics would have changed: both would still have acted in accordance with their private signals, as their aggregated beliefs are in accordance with their private ones. The only reason they where not assumed to be aggregators was to postpone the definition of aggregated beliefs.

Still Being in Cascade. Before moving on to the general model for informational cascades, two observation about agent 4 are due. First, that 4's aggregated beliefs will not be affected by the choice of 3, and second, that 4 will also be in cascade.

Preliminarily, notice that in either of the two possible initial states for 4, $\mathbf{S_4} := ((\mathbf{S_3} \otimes \mathbf{I_2}) \otimes (\mathbf{P_3}, x)) \otimes l_3$ (Fig. 9, $\alpha_3 L$ true everywhere, $x \in \{\sigma_1, \sigma_2\}$), the interpretation rules used so far will no longer be *correct*, as they reflect individualistic agents. Hence, the one-state interpretation model for 3's action should be $\mathbf{I_i}$ with $i := 3$ and precondition

$$\begin{aligned} pre(i_i) := \alpha_i L \to A_{i|\mathscr{A}} L \quad \wedge \\ \alpha_i R \to A_{i|\mathscr{A}} R. \end{aligned}$$

This interpretation model type could have been used throughout without altering the obtained due to the equivalence of 1 and 2's private and aggregated beliefs.

Using a correct interpretation rule captures that 4 (and subsequent agents) all *know* what type of agent 3 is. Interestingly, this knowledge implies that 4 will *not learn anything about 3's private beliefs from her action.* This follows as *all* states of $\mathbf{S_4}$ satisfy $\alpha_3 L \to A_{3|\mathscr{A}} L$, precisely because 3 is in cascade. Hence $\mathbf{S_4} \otimes \mathbf{I_3} = \mathbf{S_4}$.

Note that this is because 3 is in a cascade, not because she is an aggregator; had 1 and 2 been aggregators and interpreted using the new $\mathbf{I_i}$, their beliefs would still become known.

Updating $\mathbf{S_4}$ with unpointed $\mathbf{P_4}$ produces an 8 state model, see Fig. 10, as no states are deleted by the interpretation. Note, first, that at neither possible actual state ((s_1, σ_1) for private signal that L, (s_1, σ_2) for R) satisfies neither $B_4 B_3 L$ nor $B_4 B_3 R$. Hence 3 will not be counted as 'casting a vote' when determining 4's aggregated beliefs. Second, 4 knows that both 1 and 2 believe L, why, cf. the argument for 3 above, 4 will *also* be in cascade.

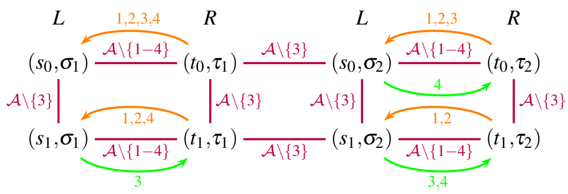


Fig. 10. The EPM $\mathbf{S}_3 \otimes \mathbf{P}_4$ without specified actual state.

5 DEL Machines and a Model of Informational Cascades

The dynamics build step-by-step may be jointly represented by a *DEL machine*. The specified machine provides a collected 'package' embodying a model of informational cascades. Hereby a formal construct is defined about which propositions may be proven.

DEL Machine. A *DEL machine* is a tuple $\mathbb{M} = \langle(\mathbf{S}_0, s_0), \mathbb{R}, \mathbb{S}, end\rangle$ where the *initial state* $(\mathbf{S}_0, s_0)$ is a pEPM, $end \in \mathscr{L}$ is the *end condition*, and $\mathbb{R}$ and $\mathbb{S}$ are (partial) functions

$$\mathbb{R} : \mathbb{N} \longrightarrow \mathscr{P}(\mathscr{R}) \text{ and}$$
$$\mathbb{S} : \mathbb{N} \longrightarrow \mathscr{P}(\mathscr{E})$$

with $\mathscr{R}$ the set of transition rules over $\mathscr{L}$ and $\mathscr{E}$ the set of pointed APMs over $\mathscr{L}$, assigning to n a *rule set* $\mathbb{R}_n$ and a *solution set* $\mathbb{S}_n$. Where $\mathbb{R}$ and $\mathbb{S}$ are partial, it is assumed that they are defined for the same initial segment of $\mathbb{N}$.

For machine $\mathbb{M}$, denote the solutions to $\mathbb{R}_n$ from $\mathbb{S}_n$ over the EPM(s) at step n by

$$next(\mathbb{M}, (\mathbf{S}, s))_n = \{(\mathbf{E}, \sigma) \in \mathbb{S}_n : (\mathbf{E}, \sigma) \text{ is a solution to all } \mathscr{T} \in \mathbb{R}_n \text{ over } (\mathbf{S}, s)\}.$$

Generating Trees. A DEL machine is run by step-wise generating a tree of pointed EPMs. Given machine $\mathbb{M} = \langle(\mathbf{S}_0, s_0), \mathbb{R}, \mathbb{S}, end\rangle$, define the tree generated by $\mathbb{M}$, $tree(\mathbb{M})$, by the following. Let the root of $tree(\mathbb{M})$ be $\mathbb{M}_0 = \{(\mathbf{S}_0, s_0)\}$. Let nodes at level $n+1$ be

$$\mathbb{M}_{n+1} = \{(\mathbf{S}, s) \otimes (\mathbf{E}, \sigma) : (\mathbf{S}, s) \in \mathbb{M}_n \text{ and } (\mathbf{E}, \sigma) \in next(\mathbb{M}, (\mathbf{S}, s))_n\}.$$

Branches are defined in a straight-forward manner: $(\mathbf{S}', s') \in \mathbb{M}_{n+1}$ is an $(\mathbf{E}, \sigma)$-successor of $(\mathbf{S}, s) \in \mathbb{M}_n$ iff $(\mathbf{E}, \sigma) \in next(\mathbb{M}, (\mathbf{S}, s))_n$ and $(\mathbf{S}', s') = (\mathbf{S}, s) \otimes (\mathbf{E}, \sigma)$. Any pointed EPM belonging to some level k such that $(\mathbf{S}, s) \models end$ is and ends it's branch and is called a *leaf*.

A properly defined DEL machines generates a tree which has a leaf at the end of each branch. It is however not hard to produce DEL machines which do not generate proper trees, e.g. if $next(\mathbb{M}, (\mathbf{S}, s))_n$ is empty for some EPM that is not a leaf.

The Machine $\mathbb{IC}$**.** Define the DEL machine $\mathbb{IC} = \langle \mathbf{S_1}, \mathbb{T}, \mathbb{S}, end \rangle$ as follows.

Initial state. Let the initial state be $\mathbf{S_1}$ (from Fig. 1, on page 5):

End condition. Let the end condition be $end := \alpha_m L \vee \alpha_m R$ with $m = max(\mathscr{A})$.

Rule and solution sets. Each agents turn consists of three steps. These are specified for n as follows. Note that the initial step of agent n's turn occurs at level $3(n-1)$, as the machine commences at level 0.

1. Interpret previous agent's action.

   $\mathbb{R}_{3(n-1)} = \{I_{n-1} = \top \rightsquigarrow [x]\top\}$

   $\mathbb{S}_{3(n-1)} = \{\mathbf{I}^{\bullet}_{n-1}\}$, the singleton APM with

$$pre(i_{n-1}) := \alpha_{n-1} L \rightarrow A_{n-1|\mathscr{A}} L \wedge \alpha_{n-1} R \rightarrow A_{n-1|\mathscr{A}} R$$

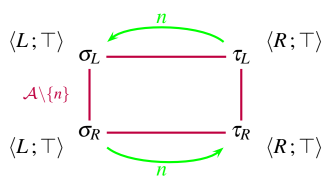
$$post(i_{n-1}) := \top.$$

   Special case: for $\mathbf{I}_0$, set $pre(i_0) = post(i_0) = \top$.

2. Receive private signal.

   $\mathbb{R}_{3(n-1)+1} = \{P_n = \top \rightsquigarrow [x]\top\}$

   $\mathbb{S}_{3(n-1)+1} = \{(\mathbf{P}_n, x_n)\}$ where $\mathbf{P}_n$ is



with x_n determined by *private signal vector* $\mathbf{P} = (x_1, x_2, ..., x_m), x_k \in \{\sigma_L, \sigma_R\}$.

3. Make choice based on aggregated beliefs.

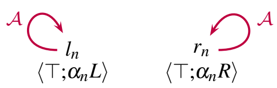
$\mathbb{R}_{3(n-1)+2} = \{\mathscr{A}_L, \mathscr{A}_R\}$:

Aggregator:

$\mathscr{A}_L = A_{i|G}L \rightsquigarrow [X]\alpha_i L$

$\mathscr{A}_R = A_{i|G}R \rightsquigarrow [X]\alpha_i R$

$\mathbb{S}_{3(n-1)+2} = \{l_n, r_n\}$:



The initial state and the end condition are self-explanatory. For the rule and solution sets, then the first two clauses ensure that interpretation models and and private signals are invoked by the 'environment' at the correct times. Further, which private signals are supplied are controlled by a vector. This vector is the interesting parameter to tweak, as it to a high degree (see Theroem 1) determines the behavior of the machine. The third clause makes agents act in a context sensitive manner, as specified by their aggregated beliefs.

Note that $\mathbb{IC}$ evolves the same irrespectively of which of the two states from $\mathbf{S}_1$ is given as actual. Let $\mathbf{P}_i$ be the private signals for agents $j < i$, i.e. the initial segment of $\mathbf{P}$ of length $i - 1$.

Given the defined machine $\mathbb{IC}$, it may be shown that it generates proper trees, i.e. that all branches are terminated by a pointed EPM that satisfies the end condition. To simply notation, note that $\mathbb{IC}$ is *deterministic*, in the sense that for every level n, the set of nodes in the generated tree, $\mathbb{IC}_n$, is a singleton. Misuse notation and refer by $\mathbb{IC}_n$ to it's element.

Proposition 1. *The machine $\mathbb{IC}$ runs until end $:= \alpha_m L \vee \alpha_m R$ is satisfied at $\mathbb{IC}_{3(m-1)+3}$, irrespectively of which initial state or which signal vector* $\mathbf{P}$ *is used for input.*

Proof. By induction it is shown that for every $i \leq m$, the machine will produce state $\mathbf{S}_{i+1}$ satisfying $\alpha_i L \vee \alpha_i R$. See the Appendix for details. **THE REMAINDER OF THE PAPER USES THE NOTATION AND SETUP FOUND N THE CONFERENCE PROCEEDINGS. I APOLOGIZE IF ANYONE ACTUALLY DOWNLOADS THIS BEFORE I GET THE REST UP TO DATE.**

'In cascade' Definition and Characterization. With $\mathscr{IC}$ defined, it is possible to precisely define the notion of *being in a cascade*: agent i is *in a cascade* iff

i) $next((\mathbf{S_i} \otimes \mathbf{I_{i-1}})) \otimes (\mathbf{P_i}, x)) = l_i$ for both $x \in \{\sigma_1, \sigma_2\}$, or

ii) $next((\mathbf{S_i} \otimes \mathbf{I_{i-1}})) \otimes (\mathbf{P_i}, x)) = r_i$ for both $x \in \{\sigma_1, \sigma_2\}$.

The definition captures that i acts in accordance with an established majority, *irrespective of her own signal.*[15]

To state the following propositions, it is handy to have notation for the agents in cascade who ignored which signals. Let $C_{Li} = \{j < i : j \text{ is in cascade and } x_j = L_j\}$ and $C_{Ri} = \{j < i : j \text{ is in cascade and } x_j = R_j\}$. We may then state and prove the following.

Lemma 1. $\mathbf{S_{n+1}} \otimes \mathbf{I_n} \models B_{n+1}B_nL \vee B_{n+1}B_nR$ *iff n is not in a cascade.*

Lemma 1 captures a crucial property regarding the higher-order reasoning occurring in cascades, namely that *the choices of agents in a cascade provide no information about their private beliefs*, i.e. private signals. The proof of the lemma and the subsequent may be found in the appendix.

Proposition 2. *If two more agents have received private signal of one type than have received signals of the other type, not counting signals of agents in a cascade, then agent i is in cascade. Precisely: if* $|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}| \geq (|\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|) + 2$ *then i is in cascade of type i), and if* $(|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}|) + 2 \leq |\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|$*, then agent i is in cascade of type ii).*

The proof rests on a counting argument. The proposition provides sufficient conditions for an agent to be in a cascade. Due to the "equivalence" of the Bayesian and aggregated beliefs approaches, these conditions are identical to those from [4], see p. 1005-06.

Corollary 1. *Cascades in $\mathscr{IC}$ are irreversible: if i is in a cascade of type i) resp. type ii), then for all $k > i$, k will be in a cascade of type i) resp. type ii).*

The corollary captures the quintessential effect of cascades, namely that they propagate through the remaining group.

Prop. 2 provides sufficient conditions for cascades to arise in $\mathscr{IC}$. The following shows that these are also necessary.

Proposition 3. *If i is in cascade, then two more agents have received private signal of one type than have received signals of the other type, not counting signals of agents*

[15] The definition thus closely mirrors that from the original paper: "An informational cascade occurs if an individual's action does not depend on his private signal." [4, p. 1000]

in a cascade. Precisely: if i is in cascade of type i), then $|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}| \geq (|\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|) + 2$*, and if i is in cascade of type ii), then* $(|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}|) + 2 \leq |\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|$.

6 Venues for Further Research

Several important aspects of informational cascades have not been discussed in this paper. First, that cascades might be *fragile*, an important focus in [4]: small changes in private signals may be sufficient to overturn even long-lived cascades. This may be investigated using variants of the system $\mathscr{IC}$, e.g. by providing agents in a cascade with hard information of the true state. It is conjectured that making *only* this change will not affect a cascade, as such a signal will not change the perceived beliefs of others. As aggregators accumulate this information, their private *knowledge* will not affect their decision. Hence, for cascades to be broken, a more detailed agent type is required who will act on private knowledge if available.

Second, aggregated belief has been defined using a self-biased tie-breaking rule, which was made "common knowledge" by the associated interpretation rule. In [4], a tie-breaking rule involving *random choice* is also used. Such a rule allows for the extraction of *less* information from the choices of others, and as noted in [10], the epistemic assumptions regarding it are non-trivial. How such a rule is to be modeled and how it affects cascades is an open question.

Finally, only a finite set of agents has been considered. Allowing an infinite set will require suitable alterations to the specified setup (e.g. EPMs must be allowed to have infinitely many states, as the state space may grow infinite due to uncertainty in cascades). From the proof of Prop. 1, it is conjectured that the infinite case of $\mathscr{IC}$ will *never* terminate. However, Prop. 2 and 3 may in this case be used to show limit results about the probability of cascades arising, and of these being correct. This requires specifying probabilities for the correctness of private signals, an aspect here ignored. It is conjectured that such results will mirror those of [4].

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Appendix: Proofs

Proposition 1. *The system $\mathscr{IC}$ runs until end $:= \alpha_m L \vee \alpha_m R$ is satisfied at $\mathbf{S_{m+1}}$, irrespectively of which initial state or which signal vector $\mathbf{P}$ is used for input.*

Proof. By induction it is shown that for every $i \leq m$, the machine will produce state $\mathbf{S_{i+1}}$ satisfying $\alpha_i L \vee \alpha_i R$.

Base case: Irrespective of the actual state chosen for $\mathbf{S_1}$, $\mathbb{T}(\mathbf{S_1}) = \{\mathscr{I}_0 = \top \rightsquigarrow [X]\top\}$ with special case $\mathbb{S}(\mathscr{I}_0) = \{\mathbf{I_0}\}$, $\mathbf{I_0}$ clearly being the unique solution to $\mathscr{I}_0$ for both initial states, why via next APM choice the system progresses to $\mathbf{S_1} \otimes \mathbf{I_0}$.

With $\mathbb{T}(\mathbf{S_1} \otimes \mathbf{I_0}) = \{\mathscr{P}_1 = \top \rightsquigarrow [X]\top\}$, the signal vector $\mathbf{P}$ is consulted, specifying either solution space $\{(\mathbf{P_1}, \sigma_1)\}$ or $\{(\mathbf{P_1}, \sigma_2)\}$, either clearly providing a unique solution to $\mathscr{P}_1$, and by next APM choice the system progresses to $(\mathbf{S_1} \otimes \mathbf{I_0}) \otimes (\mathbf{P_1}, \sigma_1)$ or $(\mathbf{S_1} \otimes \mathbf{I_0}) \otimes (\mathbf{P_1}, \sigma_2)$, satisfying resp. $A_{1|.\mathscr{A}}L$ and $A_{i|.\mathscr{A}}R$ (agent 1 is the only agent with a belief that L or that R, cf. Fig. 3 on page 7).

Whether the system progresses to one or the other, the argument is analogous, so assume the next state is $(\mathbf{S_1} \otimes \mathbf{I_0}) \otimes (\mathbf{P_1}, \sigma_1)$ for which $\mathbb{T}(((\mathbf{S_1} \otimes \mathbf{I_0})) \otimes (\mathbf{P_1}, \sigma_1)) = \{\mathscr{A}_L, \mathscr{A}_R\}$ with $\mathbb{S}(\{\mathscr{A}_L, \mathscr{A}_R\}) = \{l_1, r_1\}$. Of these two programs, only l_1 is a solution to $\{\mathscr{A}_L, \mathscr{A}_R\}$ over the given EPM, cf. the argument on page 10. Hence the system progresses to $(((\mathbf{S_1} \otimes \mathbf{I_0})) \otimes (\mathbf{P_1}, \sigma_1)) \otimes l_1 =: \mathbf{S_1}$, satisfying $\alpha_1 L$ so also $\alpha_1 L \vee \alpha_1 R$, which concludes the base case.

Inductive step: Assume the system has reached state $\mathbf{S_n}$, $n < m$, satisfying $\alpha_{n-1}L \vee \alpha_{n-1}R$. It is shown the system then progresses to $\mathbf{S_{n+1}}$, satisfying $\alpha_{n+1}L \vee \alpha_{n+1}R$. For $\mathbf{S_n}$, $\mathbb{T}(\mathbf{S_n}) = \{\mathscr{I}_{n-1} = \top \rightsquigarrow [X]\top\}$ with $\mathbb{S}(\{\mathscr{I}_{n-1}\}) = \{\mathbf{I_{n-1}}\}$, $\mathbf{I_{n-1}}$ clearly being the unique solution to $\mathscr{I}_{n-1}$. Hence the system progresses by next APM choice to $\mathbf{S_n} \otimes \mathbf{I_{n-1}}$.

With $\mathbb{T}(\mathbf{S_n} \otimes \mathbf{I_{n-1}}) = \{\mathscr{P}_n = \top \rightsquigarrow [X]\top\}$, the signal vector $\mathbf{P}$ is consulted, specifying either solution space $\{(\mathbf{P_n}, \sigma_1)\}$ or $\{(\mathbf{P_n}, \sigma_2)\}$, either clearly providing a unique solution to $\mathscr{P}_n$, and by next APM choice the system progresses to $(\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, \sigma_1)$ or $(\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, \sigma_2)$. Both states will satisfy $A_{n|.\mathscr{A}}L$ or $A_{n|.\mathscr{A}}R$, depending not only on the private signal just invoked, but also on whether the agent is in a cascade or not. However, given the definition of $A_{i|.\mathscr{A}}$, it is clear that an for EPM satisfy $\neg(A_{i|.\mathscr{A}}L \vee A_{i|.\mathscr{A}}R)$, it must also satisfy $\neg(B_i L \vee B_i R)$ as the aggregated belief will otherwise be determined either by majority or tie-breaking. But given the last update with $(\mathbf{P_n}, \sigma_1)$ or $(\mathbf{P_n}, \sigma_2)$, agent n will privately believe L or R, why either $A_{n|.\mathscr{A}}L$ or $A_{n|.\mathscr{A}}R$ will be satisfied.

This in turn guarantees that one of the transition rules from $\mathbb{T}(((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes (\mathbf{P_n}, x)) = \{\mathscr{A}_L, \mathscr{A}_R\}$ will be active. As $\mathbb{S}(\{\mathscr{A}_L, \mathscr{A}_R\}) = \{l_n, r_n\}$ clearly provides a unique solution to $\{\mathscr{A}_L, \mathscr{A}_R\}$ over both EPMs in question, the system will progress to either of four states (two options for each signal, as the agent may be in a cascade): $((\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, \sigma_1)) \otimes l_n$ or $((\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, \sigma_1)) \otimes r_n$ or $((\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, \sigma_2)) \otimes r_n$ or $((\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, \sigma_2)) \otimes r_n$. By the postconditions of l_n and r_n, each will satisfy

$\alpha_n L \vee \alpha_n R$. Given that $\mathbf{S_{n+1}} := (((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes \mathbf{P_n}) \otimes next(((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes \mathbf{P_n})$, $\mathbf{S_{n+1}}$ satisfies $\alpha_n L \vee \alpha_n R$, which concludes the inductive step.

Lemma 1. $\mathbf{S_{n+1}} \otimes \mathbf{I_n} \models B_{n+1}B_n L \vee B_{n+1}B_n R$ *iff n is not in a cascade.*

Proof. No matter the actual state of $\mathbf{P_n}$ $(x \in \{\sigma_1, \sigma_2\})$, no matter what action $(y \in \{l_n, r_n\})$, the model $((\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, x)) \otimes y$ will contain two parts disconnected for n; one in which $B_n L$ and one where $B_n R$. These are connected for all $\mathscr{A} \backslash \{n\}$. The only way to obtain either $B_{n+1}B_n L$ or $B_{n+1}B_n R$ from this via $\mathbf{I_n}$ is if one of these parts are deleted. A state will be deleted by/not survive update with $\mathbf{I_n}$ iff it does not satisfy $pre(i_n) = \alpha_n L \rightarrow A_{n|\mathscr{A}} L \wedge \alpha_n R \rightarrow A_{n|\mathscr{A}} R$. Hence we must show that an agent is in a cascade *iff* all states of models $((\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes (\mathbf{P_n}, x)) \otimes y$ satisfy $pre(i_n)$.

Take the case of l_n (r_n is symmetrical), and regard the unpointed model $((\mathbf{S_n} \otimes \mathbf{I_{n-1}}) \otimes \mathbf{P_n}) \otimes l_n$. Then showing *satisfaction of* $pre(i_n)$ *reduces to showing satisfaction of* $A_{n|\mathscr{A}} L$: In all states $\alpha_n L$ is satisfied, so 1) $\alpha_n R \rightarrow A_{n|\mathscr{A}} R$ is trivially satisfied, and 2) in all states $\alpha_n L \rightarrow A_{n|\mathscr{A}} L$ iff $A_{n|\mathscr{A}} L$ in all states.

Both ways: Assume, matching the l_n case, that n is in a cascade of type *i*), i.e. that $next((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes (\mathbf{P_n}, x)) = l_n$ for both $x \in \{\sigma_1, \sigma_2\}$. Given the aggregator decision rules, this clearly occurs *iff* $A_{n|\mathscr{A}} L$ is satisfied in the two possible actual states of $(\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes (\mathbf{P_n}, x)$. This happens *iff* $A_{n|\mathscr{A}} L$ is satisfied in all this model's states, again *iff* $A_{n|\mathscr{A}} L$ is satisfied in all states of $((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes (\mathbf{P_n}, x)) \otimes l_n$ (as l_n does not change this).

Proposition 2. *If two more agents have received private signal of one type than have received signals of the other type, not counting signals of agents in a cascade, then agent i is in cascade. Precisely: if* $|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}| \geq (|\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|) + 2$ *then i is in cascade of type i), and if* $(|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}|) + 2 \leq |\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|$*, then agent i is in cascade of type ii).*

Proof. Shown for the type *i*) case only, as the other is symmetrical. Assume that $|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}| \geq (|\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|) + 2$. It follows that at $\mathbf{S_i}$, $|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}|$ will have acted according to their private signal that L without being in a cascade, and $|\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|$ have done the same for signal R. Hence, following interpretation, $|\{j : \mathbf{S_i} \otimes \mathbf{I_{i-1}} \models B_i B_j L\}| \geq |\{j : \mathbf{S_i} \otimes \mathbf{I_{i-1}} \models B_i B_j R\}| + 2$, as the private beliefs of non-cascading agents are revealed by the interpretation model (Lemma 1).

To abbreviate, let $L_{SI} = \{j : \mathbf{S_i} \otimes \mathbf{I_{i-1}} \models B_i B_j L\}$ and $L_{SIP} = \{j : (\mathbf{S_i} \otimes \mathbf{I_{i-1}}) \otimes (\mathbf{P_i}, x) \models B_i B_j L\}$ with $x \in \{\sigma_1, \sigma_2\}$ specified by context, and let R_{SI} and R_{SIP} be the $B_i B_j R$-counterparts.

If i receives private signal that L by $(\mathbf{P_i}, \sigma_1)$, then clearly $|L_{SIP}| > |L_{SI}|$, why $(\mathbf{S_i} \otimes \mathbf{I_{i-1}}) \otimes (\mathbf{P_i}, \sigma_1) \models A_{i|\mathscr{A}} L$. So the next APM choice will be l_i.

If i receives private signal that R by $(\mathbf{P_i}, \sigma_2)$, then $|L_{SIP}| = |L_{SI}|$ while $|R_{SIP}| = |R_{SI}| + 1$. Hence $|L_{SIP}| \geq |R_{SIP}| + 1$. As the tie-breaking parameters takes value at most ½, we have that $|L_{SIP}| > |R_{SIP}| + ½$, why $(\mathbf{S_i} \otimes \mathbf{I_{i-1}}) \otimes (\mathbf{P_i}, \sigma_2) \models A_{i|\mathscr{A}} L$. So the next APM choice will be l_i. Conjoining the two cases shows that i is in a cascade of type i).

Proof of Proposition 3. *If i is in cascade, then two more agents have received private signal of one type than have received signals of the other type, not counting signals of agents in a cascade. Precisely: if i is in cascade of type i), then* $|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}| \geq (|\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|) + 2$*, and if i is in cascade of type ii), then* $(|\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}|) + 2 \leq |\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|$.

Proof. Shown for the type i) case only, as other is symmetrical. Assume i is in a cascade of type i) so $next((\mathbf{S_i} \otimes \mathbf{I_{i-1}})) \otimes (\mathbf{P_i}, x)) = l_i$ for both $x \in \{\sigma_1, \sigma_2\}$. This occurs iff $(\mathbf{S_i} \otimes \mathbf{I_{i-1}})) \otimes (\mathbf{P_i}, x) \models A_{i|\mathscr{A}} L$ for both $x \in \{\sigma_1, \sigma_2\}$ (if $(\mathbf{S_i} \otimes \mathbf{I_{i-1}})) \otimes (\mathbf{P_i}, \sigma_2) \models A_{i|\mathscr{A}} R$, the next APM choice would be r_i). From satisfied $A_{i|\mathscr{A}} L$, it follows that $\alpha + |L_{SIP}| > \beta + |R_{SIP}|$.[16] For $x = \sigma_2$, $\alpha = 0$ and $\beta = ½$, so $|L_{SIP}| > |R_{SIP}| + ½$. Further, as i adds 1 to $|R_{SIP}|$ due to private signal, $|L_{SI}| > |R_{SI}| + 1½$. As only agents not in a cascade add to $|L_{SI}|$ and $|R_{SI}|$, each will have followed their private signals, why $|L_{SI}| = |\{j \in \mathscr{A} : L_j \in \mathbf{P}_i\}| - |C_{Li}|$ and $|R_{SI}| = |\{j \in \mathscr{A} : R_j \in \mathbf{P}_i\}| - |C_{Ri}|$, and the conclusion follows. The argument for $x = \sigma_1$ is analogous.

[16] See the proof of Prop. 2 for definitions of L_{SIP}, R_{SIP}, L_{SI} and R_{SI}.