Hintikka's *Knowledge and Belief* in Flux

Vincent F. Hendricks[1] and Rasmus K. Rendsvig[1,2]

[1] CIBS, Center for Information and Bubble Studies, University of Copenhagen
[2] LUIQ, Theoretical Philosophy, Lund University

Abstract. Hintikka's *Knowledge and Belief* from 1962 is considered the seminal treatise on epistemic logic. It provides the nuts and bolts of what is now a flourishing paradigm of significance to philosophy, economics, mathematics and theoretical computer science — in theory as well as practice. And in theory and for practice epistemic logic has been extensively articulated, refined and developed especially with respect to capturing the dynamics of reasoning about knowledge. But although the robust narrative about Hintikka's epistemic logic is rather static, the leap to dynamic epistemic logic is right there back in 1962 as this paper will show.

Keywords: Jaakko Hintikka, epistemology, formal epistemology, epistemic logic, dynamic epistemic logic, public announcement, arbitrary announcement

Note: This is a penultimate draft of the published version. The differences are all minor.

1 Dynamics of the Seminal

The development of epistemic logic, was, according to Hintikka, not to be judged by its own technical merits. Epistemic logic was to be assessed on its ability to inform and align epistemology — the logical exercises and insights were intended to say something about the rationality of inquiry and what it *means* to know, not what it *is* to know. Hintikka categorically considered the latter attempt of defining knowledge as an exercise in futility, as is evident in his 2007 "Epistemology without Knowledge and without Belief" [17]. The point of departure is the logic of expressions involving knowledge, belief and other propositional attitudes and in this way epistemic logic *a la* Hintikka is a precursor to much formal epistemology worrying less about a proper definition of knowledge and more about how information behaves and what you can do with it on your own and in groups [2].

What you can do with knowledge and belief alone — or between agents reasoning or acting in concert — invokes studying the dynamics of such notions as opposed to the statics of knowledge possession and justification. Today dynamic epistemic logic is a potent tool for modelling a variety of themes ranging from distributed and common knowledge, public and private announcements, agent interaction, social influence, interrogative inquiry, strategies in games, epistemic conditions of bilateral trade all the way to no trade theorems; themes none of which are addressed in *Knowledge and Belief* [16]. But all the same there is a dynamic trait to Hintikka's thinking about the logic of knowledge and belief. In fact, much of what he has to say about the logic of the two notions would make little sense if Hintikka was not on the move and, as a matter of fact, on par with the *logic of arbitrary public announcements*.

2 Committing to "I know that φ"

To say "I know that φ" is not a statement to be uttered lightly —- you commit to quite a bit asking Hintikka: Not only do you commit yourself to the truth of φ, but also to being in an evidential situation with conclusive grounds strong enough to warrant the claim. What constitutes conclusive grounds is not strictly defined.[3] Be that as it may, Hintikka's subsequent analysis feeds strongly on one necessary consequence of such grounds:

> If somebody says "I know that *p*" in this strong sense of knowledge, he implicitly denies that any further information would have led him to alter his view. He commits himself to the view that he would still persist in saying that he knows that *p* is true — or at the very least persist in saying that *p* is in fact true — even if he knew more than he now knows. [16, p. 18a]

This consequence of the speaker's commitment lies at the heart of Hintikka's analysis, and throughout *Knowledge and Belief* he routinely returns to re-castings of it. In particular, it is decisive for the nature of an *epistemic alternative*:

> The conditions into which we are trying to catch the logic of knowledge and belief are in terms of certain alternatives to a given state of affairs. Roughly

[3] On this point, Hintikka takes a thoroughly pragmatic view: "We must realize, however, that having this right [to claim knowledge] need not mean that one's grounds are so strong that they logically imply that what one claims to know is true.[5] It may merely mean that the grounds one has are such that any further inquiry would be pointless for the normal purposes of the speakers of the language.[6]" [16, p. 17-18]. Footnotes are references to resp. N. Malcolm and D. Arner.

> speaking, these alternatives are possible states of affairs in which a certain person knows at least as much as — and usually even more than — he knows in the given state. In short, we are concerned with the different possibilities there are for somebody to gain further information. [16, p. 44][4]

Hence, the commitment on the side of the speaker is one relating his current informational state to other such states — in essence, those in which the speaker has the same or more information. A speaker committing to his epistemic attitude will accordingly be able to survive a particular type of *information change* from potentially one state to another. Here is the strong dynamic flavor of the very account Hintikka advances back in 1962.

To put the epistemological program front and center, Hintikka stipulated that the axioms or principles of epistemic logic, all but too familiar today, are really conditions describing a special kind of general (strong) rationality. The statements which may be proved false by application of the epistemic axioms are not inconsistent in the sense that their truth is logically impossible. They are rather rationally 'indefensible'. Indefensibility is annexed as the agent's epistemic laziness, sloppiness or perhaps cognitive incapacity whenever to realize the implications of what he in fact knows:

> In order to see this, suppose that a man says to you, 'I know that p but I don't know whether q' and suppose that p can be shown to entail logically q by means of some argument which he would be willing to accept. Then you can point out to him that what he says he does not know is already implicit in what he claims he knows. If your argument is valid, it is irrational for our man to persist in saying that he does not know whether q is the case. [16, p. 31]

Defensibility thus means not falling victim of 'epistemic neglience' [10]. The notion of indefensibility gives away the status of the epistemic axioms and logics embraced by Hintikka in *Knowledge and Belief*. Some epistemic statement for which its negation is indefensible is called 'self-sustaining'. The notion of self-sustenance corresponds to the concept of validity. Corresponding to the self-sustaining statement is the logically valid statement. This will in turn be a statement which is rationally indefensible to deny. In conclusion, epistemic axioms are descriptions of rationality. Thus epistemic logic is informing and aligning epistemology. Still, this doesn't wrap up fleshing out

[4] In reading the latter quote, the formulation "knows at least as much" should not be taken as meaning that the agent can know "as much" (given some measure), but possibly different; rather, the agent should know the same or more.

the dynamics of Hintikka's account to which attention is turned next and for the remainder of this paper.

3 The Mathematics of *Knowledge and Belief*

The syntax and semantics introduced by Hintikka includes both knowledge and belief operators, but to limit the exposition, attention is restricted to the knowledge part of the story. Moreover, the exposition is limited to the single agent case.

Apart from regular Boolean connectives, Hintikka introduces two epistemic operators. Where a is an agent and φ a formula, so are

$$K_a\varphi \text{ and } P_a\varphi.$$ [5]

The intended reading of $K_a\varphi$ is "a knows that φ" and $P_a\varphi$ "it is possible, for all a knows, that φ".

Syntactically, there is not that much more to it, but semantically the plot of course thickens. Although, some 40 years down the line, Hintikka maintains that it is not too complicated after all, at least not for the way he envisioned epistemic logic:

> What the concept of knowledge involves in a purely logical perspective is thus a dichotomy of the space of all possible scenarios into those that are compatible with what I know and those that are incompatible with my knowledge. This observation is all we need for most of epistemic logic. [17, p. 12]

To model the notion of epistemic alternatives — those scenarios that are compatible with current information — and provide formal arguments for the self-sustainability of various principles of epistemic logic, Hintikka introduces the notion of a *model system*, each such object consisting of a set of *model sets* related by an *alternativeness relation*.

3.1 Model Sets

The brass tacks consists of the (partial) description of a state of affairs captured by a *model set*: A set μ of sentences closed under the following rules

(C.¬) If $\varphi \in \mu$, then not $\neg\varphi \in \mu$.

[5] We use modern, standardized notation. Where Hintikka writes "$K_a p$"$\in \lambda$, we omit the quotes and where Hintikka uses lowercase letters for arbitrary formulae, we use φ, ψ, reserving lowercase latins for atomic propositions.

(C.∧) If $\varphi \wedge \psi \in \mu$, then $\varphi \in \mu$ and $\psi \in \mu$
(C.∨) If $\psi \vee \psi \in \mu$, then $\varphi \in \mu$ or $\psi \in \mu$ (or both).
(C.¬¬) If $\neg\neg\varphi \in \mu$, then $\varphi \in \mu$.
(C.¬∧) If $\neg(\varphi \wedge \psi) \in \mu$, then $\neg\varphi \in \mu$ or $\neg\psi \in \mu$ (or both).
(C.¬∨) If $\neg(\varphi \vee \psi) \in \mu$, then $\neg\varphi \in \mu$ and $\neg\psi \in \mu$.

A set of propositional logical formulas is consistent *iff* it can be embedded in a model set, though model sets need not be maximal. For one, a model set need not contain literals. Further, the rules above require only that model sets are closed suitably under subformulas; they do not require, e.g., that model sets are closed under introduction of disjunction. Model sets thus allow for the representation of partial states (as noted in [1]).

3.2 Model Systems

Keeping the idea of embeddability of defensible (consistent) sets of formulas central, Hintikka generalizes the notion of a model set to that of a *model system* in order to accommodate formulas with epistemic operators.

A (single agent) model system is a pair (Ω, R_a) where Ω is a set of model sets satisfying (C.¬K), (C.¬P) and (C.K) immediately below, and R_a a binary relation on Ω for agent a. R_a is called the *relation of alternativeness* (p. 35) for a. It must so be that the model system satisfies the criteria (C.P*) and (C.KK*) below.

Of the three additional requirements on model sets, the first two provide a weak version of duality of the possibility and knowledge operators:

(C.¬K) If $\neg K_a\varphi \in \mu$, then $P_a\neg\varphi \in \mu$.
(C.¬P) If $\neg P_a\varphi \in \mu$, then $K_a\neg\varphi \in \mu$.

The third uncontroversially require that knowledge is truthful:

(C.K) If $K_a\varphi \in \mu$, then $\varphi \in \mu$.

The two conditions which the alternativeness relations satisfy capture a very central feature of Hintikka's use of the epistemic operators. Jointly, they ensure that a formula $P_a\varphi$ involving the epistemic possibility operator captures its intended meaning: *it is possible, for all that a knows, that* φ. Condition (C.P*) captures that φ is indeed possible by requiring that there exists at least one epistemic alternative in which φ is the case. The condition (C.KK*) restricts the type of such alternatives to fit Hintikka's "no further information would make the speaker deny the claim to knowledge"-viewpoint. This is done by requiring that the set of known formulas is weakly increasing along the alternativeness relation:

(C.P*) If $P_a\varphi \in \mu$ for $\mu \in \Omega$, then there is an a-alternative μ^* to μ such that $\varphi \in \mu^*$.

(C.KK*) If $K_a\varphi \in \mu$ and μ^* is an a-alternative to μ, then $K_a\varphi \in \mu^*$.

Or in Hintikka's own words:

> The condition (C.P*) serves to make sure that it is possible that p. We required more, however; we required that it is possible, for all that the person referred to by the term a knows, that p. Hence everything he knows in the state of affairs described by μ, he also has to know in the alternative state of affairs described by μ^*. In other words, the [(C.KK*)] condition has to be imposed on the model sets of a given model system (p. 35)

Given the model system construction, *defensibility* of a set of sentences is defined as its ability to be embedded in a model set of a model system.

Hintikka relays that the criteria (C.P*) and (C.KK*) are equivalent to assuming the alternativeness relation to be *reflexive* and *transitive* together with

(C.K*) If $K_a\varphi \in \mu$ and if $\mu R_a \mu^*$, then $\varphi \in \mu^*$.

He does argue against the relation being necessarily symmetric, hereby distinguishing the nature of the relation from that often assumed nowadays in mainstream epistemic logic. The argument presented runs accordingly:

> For this purpose, let us recall that a model set μ_2 is an alternative to μ_1 if, and only if, intuitively speaking, there is nothing about the state of affairs described by the former that is incompatible with what someone knows in the state of affairs described by the latter. Now it is obviously not excluded by what I now know that I should know more than I now do. But such additional knowledge may very well be incompatible with what now is still possible, as far as I know. (p. 35)

An alternative argument to the same effect is that acquiring additional information would otherwise become impossible: if $\mu_1 R_a \mu_2$, then a has the same or more information available in μ_2 than in μ_1. Hence, if also $\mu_2 R_a \mu_1$, then a must have the same information in μ_1 and μ_2. If the relation was assumed symmetric, each connected component of the model system would collapse to an equi-informed partition cell — exactly as assumed in modern mainstream epistemic logic. More on comparing the two approaches in below.

4 Epistemic Logic and Kripke Models

Hintikka's model system semantics for epistemic logic may be related in a multitude of ways to the mainstream Kripke model semantics. Here is one: Given a finite, non-empty set of propositional atoms Φ and a single agent, a, a (single agent) *Kripke model* is a tuple $M = (\llbracket M \rrbracket, \sim_a, \llbracket\cdot\rrbracket_M)$ where

> $\llbracket M \rrbracket$ is a non-empty set of *states*,
>
> $\sim_a \subseteq \llbracket M \rrbracket \times \llbracket M \rrbracket$ is an accessibility relation, and
>
> $\llbracket\cdot\rrbracket_M : \Phi \longrightarrow \mathscr{P}(\llbracket M \rrbracket)$ is a valuation map assigning to each atom $p \in \Phi$ an extension $\llbracket p \rrbracket_M \subseteq \llbracket M \rrbracket$.

The subscript of $\llbracket\cdot\rrbracket_M$ is omitted when clear from context.

Formulas are evaluated relative to a *pointed* Kripke model, a pair (M,s) where $s \in \llbracket M \rrbracket$. (M,s) is also written Ms. Letting K_a be the (normal modal) knowledge operator of agent a, semantics of modal formulas are given by

$$Ms \models K_a\varphi \text{ iff for all } t \text{ such that } s \sim_a t,\ Mt \models \varphi.$$

The remaining formulas have standard semantics, with P_a being the dual operator of K_a. The set of states in M that satisfy φ is given by $\llbracket \varphi \rrbracket_M := \{t \in \llbracket M \rrbracket : Mt \models \varphi\}$.

4.1 Model Systems and Kripke Models

A straightforward manner of relating Hintikka's epistemic logic and semantics to Kripke models and their associated logics is to transform Hintikka's model systems into models for the logic **S4**, i.e. Kripke models with a reflexive and transitive accessibility relation (see e.g. [9]).

Given a model system (Ω, R_a), a Kripke model $(\llbracket M \rrbracket, \sim_a, \llbracket\cdot\rrbracket_M)$ is constructable. First, for every epistemic alternative μ in Ω, let $\llbracket M \rrbracket$ contain a state s_μ, and let $\llbracket M \rrbracket$ contain no further states. Second, let the relation $\sim_a$ be given by the relation R_a, i.e. let $s_\mu \sim_a s_{\mu^*}$ if, and only if, $\mu R_a \mu^*$. Third, for every atomic proposition $p \in \Phi$, let the valuation of p be given by $\llbracket p \rrbracket_M := \{s_\mu \in \llbracket M \rrbracket : p \in \mu\}$.

The first two requirements ensure that the frame of the resulting Kripke model is isomorphic to the frame of the model system. There is a design choice to be made while constructing the valuation. Now $\llbracket\cdot\rrbracket_M$ is a total map, ensuring, by the standard semantics, that either $Ms \models p$ or $Ms \models \neg p$.[6] This is not the case while relying on the

[6] The interested reader is referred to [1] for a Kripke model-style construction that does have this property.

model systems' semantics. A model set $\mu \in \Omega$ need not be complete with respect to the set of atomic propositions Φ. That is, although $p \in \Phi$, possibly $p \notin \mu$ and $\neg p \notin \mu$.

A consequence of this difference is that the notion of knowledge modelled using Kripke semantics is somewhat stronger than the one obtained using model systems semantics. In particular, the former notion of knowledge is a *logically omniscient* one. In turn, although $K_a\varphi \in \mu$ in (Ω, R_a) implies $Ms_\mu \models K_a\varphi$, the converse implication does not hold.

All the same the two semantics agree on the validity of important epistemic principles **T**, i.e. $K_a\varphi \to \varphi$, and **4**, i.e. $K_a\varphi \to K_aK_a\varphi$. And, all the same, there are (at least) three important differences. The first is related to the problem of logical omniscience, a theme which will be repressed. The second is technical and pertains to the ordering of epistemic alternatives according to how informed the agent is. The third is interpretational and relates to the notion of positive introspection.

4.2 Kripke Model States lack Informational Structure

The title of this section encapsulates the technical difference: Kripke model states lack the necessary information structure to properly fill the role of Hintikka's epistemic alternatives. The argument for this postulate is simple:

Take two model sets μ and μ^* from a model system. Then each will contain a subset of knowledge formulas. Depending on the nature of these knowledge formula subsets, μ and μ^* may be related by the alternativeness relation in accordance with Hintikka's requirements: If the agent knows more in μ^* than in μ, then μ^* is an epistemic alternative to μ and *vice versa*. If neither is compatible with the information possessed by the agent in the other, then they will be unrelated. Hence the internal structure of the model sets induce the relation of alternativeness through Hintikka's technical requirements. This is also the reason why Hintikka always maintained that the epistemic accessibility relation is the most basic of all as it relates directly to both the partition of states in accordance with the cognitive attitude (and those not) and the internal informational structure of model sets. Alternativeness relations are neither primitive nor complex beyond conceptual capture, they are based on information structure and vocational requisites — it's as simple as that.

For two states from a Kripke model, no counterpart operation for constructing the alternativeness relation exists. The states do not have a rich enough internal structure to facilitate a comparison. Both are possible world descriptions, but they only describe, through their valuation, the basic ontic facts of postulated worlds.

Isolated, such states do not describe the agent's information. Hence these states do not represent anything that allows for an ordering by informational content.

Though reflexive and transitive Kripke models may yield a logic closely related to Hintikka's, and though the relation in such models may be interpreted as one relating epistemic alternatives, this interpretation is, from a formal point of view, void: Kripke model states lack the necessary informational structure to induce the alternativeness relation.

4.3 Knowing that One Knows and Positive Introspection

In most renditions of epistemic logic, one will find **4**, i.e. $K_a\varphi \rightarrow K_aK_a\varphi$, referred to as the *axiom of positive introspection*. This label captures the idea that knowledge as a mental state is transparent to the mind's eye. Upon reflection, one will recognize one mental state as one of knowing the φ in question.

Hintikka's model systems and reflexive-transitive Kripke models both validate **4**, but Hintikka does not support the philosophical thesis of positive introspection. In fact, it is argued at length in *Knowledge and Belief* that arguments from introspection are invalid when they pertain to the type of knowledge Hintikka seeks logic for. Moreover, **4** indeed follows as a theorem from Hintikka's framework but justified without treating knowledge as a mental state and hence without reference to introspection. The reason for celebrating **4** is logical rather than psychological. **4** follows as a theorem since the relation between model sets is transitive when induced by the amount of information possessed. That's logical, not psychological and hence not a question of introspection.

Albeit Hintikka is unambiguous about his intent of not capturing the logic of knowledge introspection, he does allude to principles that introspective knowledge adheres to. It seems as if Hintikka would support that introspective knowledge does satisfy both positive and negative introspection (i.e., **4** and **5**: $\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$). To wit, he finds that positive and negative introspection go hand in hand:

> Although the intimations of the argument from introspection are not without substance, they are entirely fallacious when taken at their face value [and applied to the type of knowledge Hintikka is interested in]. This is betrayed, among other things, by the fact that they prove far too much [about said knowledge type]. If they are right, clearly they must work both ways. If I can find out by searching my mind what I know or what I believe, I must similarly be able to find out what I do not know or what I do not believe. (This has in effect been claimed by some people who rely on introspective

> arguments.) In other words, it ought to follow, *inter alia*, that whenever I do not know something, I virtually know that I do not know it. [16, p. 42]

On top, he argues that introspective knowledge is fully transparent:

> Now it is characteristic of the introspective knowledge we have of our own mental states that there is no room for further information. If something can only be known to me by introspection, then almost *per definitionem* I know all there is to be known about it; the notion of having further evidence becomes empty. [16, p. 44]

That an introspective mental state leaves no room for further information seems to be in accordance with the acceptance of **5**: If an internal mental state is also modeled using some notion of states, then "no further information" would seem to entail that whatever states are deemed compatible with current evidence must also be judged as equally informative. Hence, if a relation was to be introduced on such states, that relation would be an equivalence relation, and the introspective knowledge would thus satisfy **5**, a principle Hintikka rejects for a notion of "ordinary" knowledge, but all the same might accept for a concept of knowledge of an introspective nature:

> I shall not criticize this line of thought nor the notion of knowledge by introspection. The logic of introspection is likely to have its peculiarities.
>
> [16, p. 44]

4.4 S5, Introspection and Epistemic States

The **S5** type of knowledge, governed by axioms **T**, **4** and **5**, is today, give or take, *the* modelling standard in the epistemic logic community. Where the accessibility relation is an equivalence relation, a pointed Kripke model is often called an *epistemic state*. If states s and t are related by $\sim_a$ in an epistemic state, the canonical interpretation is that the agent possesses *exactly the same information* in the two states, i.e., they are *indistinguishable* to a.

Though both **4** and **5** are referred to as principles of introspection, the philosophical contestation of knowledge as an internal mental state need however not be accepted at face value. Alternatively, epistemic states and their logic may be read as strong rationality criteria for agents that reflect upon the evidence they have received so far and what that evidence rules out right around now; agents who knows what the remaining possibilities are — but whatever they are, none of it hinges on

5 being introspective or psychological. If anything, it is more like a closed world assumption in force [15].

This interpretation of **S5** and its models do seem congruent with Hintikka's project. Not as a description of the principles of knowledge however, but as a characterization of the structure of a *static* informational context; one in which all options have been considered, some eliminated by evidence, and some actively under consideration. Indeed, Jaakko Hintikka did like a good mystery and detective fiction:

> How often have I said to you that when you have eliminated the impossible, whatever remains, *however improbable*, must be the truth?
>
> Sherlock Holmes
> (Sir A. C. Doyle, *The Sign of The Four*)

On this understanding of the things to be, epistemic states, governed by an **S5** logic, seem properly to play the role of epistemic alternatives. Formally they prevail where Kripke model states do not. Epistemic states *do* allow for a straightforward ordering in terms of amounts of information retained, from which the alternativeness relation again may be induced.

5 Epistemic Alternatives as Epistemic States

An epistemic state Ms for single agent a directly reflect the amount of information possessed by the agent.[7] The amount of information is given by the agent's *range of uncertainty*: The smaller the range of uncertainty, the more informed the agent is. The range of uncertainty is directly represented by the set of states in Ms: The more states in $[\![M]\!]$, the more possibilities the agent considers possible, so the less informed the agent is.

Formally, the epistemic states in the set $\mathbf{M}_\Phi$ of epistemic states based on atomic propositions Φ may be partially ordered in accordance with the amount of information possessed by subset inclusion on the models' domains. For epistemic states Ms and Nt, if $[\![M]\!] \subseteq [\![N]\!]$, then the agent is at least as informed in Ms as she is in Nt.[8]

[7] Technically, identify Ms with it's *generated submodel rooted at* s*'s bisimulation quotient*, see [14], Sec. 3.6. This submodel is identical with a's *information cell* relative to s in Ms, i.e. $\sim_a(s) = \{t \in [\![M]\!] : s \sim_a t\}$ after the elimination of states with identical valuations (which are redundant for single agent **S5**). The states in $[\![M]\!]$, but not in the information cell are, from modal logical point of view, superfluous. Given the set Φ of proposition symbols is finite, each reduced information cell is finite.

[8] Cf. footnote 7 and identifying states with identical valuations.

Subset inclusion yields a partial order $\succeq$ on the set $\mathbf{M}_\Phi$ by $Nt \succeq Ms$ iff $[\![N]\!] \subseteq [\![M]\!]$. The order sign $\succeq$ is reversed when compared to the inclusion in order to reflect that the agent is more informed in Nt than in Ms. This is also neatly reflected by the fact that $Ms \preceq Nt$ implies that for non-epistemic φ, $Ms \models K_a\varphi$ implies $Nt \models K_a\varphi$.

Taking the partial order $\preceq$ as constituting the alternativeness relation, a notion of epistemic alternative in the spirit of Hintikka's may be defined on the set of epistemic states $\mathbf{M}_\Phi$:

For agent a, Ms has as an epistemic alternative Nt *iff* $Ms \preceq Nt$.

From this definition, the epistemic state-based version of a model system follows obviously: The model system based on Φ is the pair $(\mathbf{M}_\Phi, \preceq)$.

5.1 Knowledge Operators

In ballpark terms, the semantics of the Hintikkian knowledge operator K_a relative to an epistemic state Ms in some model system $(\mathbf{M}_\Phi, \preceq)$ are to capture that something is the case in all epistemic alternatives to Ms. Given that each epistemic state is itself a Kripke model, two types of operators are habitual to consider: One capturing dynamics, Hintikkian knowledge and possibility, K_a and its dual P_a, and one commandeering the static knowledge about current informational context, k_a. For atoms $p \in \Phi$, the following language will be serviceable:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid k_a\varphi \mid K_a\varphi \mid P_a\varphi$$

The model system semantics for the static operator k_a (with dual $p_a := \neg k_a \neg$) are normal modal logical semantics. They are defined using the feature of model systems in which every epistemic state is a Kripke model. Let $\Omega = (\mathbf{M}_\Phi, \preceq)$ be a model system and let $Ms = ([\![M]\!], \sim_a, [\![\cdot]\!]_M, s) \in \mathbf{M}_\Phi$. Then

$$\Omega, Ms \models \varphi \text{ iff } Ms \models \varphi, \text{ for non-modal } \varphi.$$
$$\Omega, Ms \models k_a\varphi \text{ iff } \forall t \sim_a s : \Omega, Mt \models \varphi.$$

As Ω is also a Kripke frame — notwithstanding informationally structured states — a similar clause bequeath the semantics for the K_a operator:

$$\Omega, Ms \models K_a\varphi \text{ iff } \forall Nt \in \mathbf{M}_\Phi : \text{ if } Ms \preceq Nt, \text{ then } \Omega, Nt \models \varphi.$$

Finally, the P_a operator's semantics bank on the existence of an epistemic alternative:

$$\Omega, Ms \models P_a\varphi \text{ iff } \exists Nt \in \mathbf{M}_\Phi : Ms \preceq Nt \text{ and } \Omega, Nt \models \varphi.$$

Accordingly defined, the semantics capture the ballpark description above. Moreover, they fit the bill when dining at Hintikka's.

5.2 Properties of Knowledge in Model Systems

Model systems, and in particular $\preceq$, have been defined using a natural measure of information inherent in **S5** models. This is reflected in the properties of the K_a operator, which satisfies Hintikka's requirements.

First, the weak duality principles (C.¬K) and (C.¬P) on p. 5 are satisfied by the validity of $P_a\varphi \leftrightarrow \neg K_a\neg\varphi$, trivially established by the semantic definitions. This validity is stronger than required by Hintikka as it also yields e.g. $P_a\neg\varphi \to \neg K_a\varphi$. It seems reasonable to assume that Hintikka would accept this.

A question arises as to whether the defined semantics yield a system stronger than Hintikka would be buying, notably pertaining to the behavior of the P_a operator. Hintikka requires

(C.P*) If $\Omega, Ms \models P_a\varphi$, then $\exists Nt \in \mathbf{M}_\Phi$ such that $Ms \preceq Nt$ and $\Omega, Nt \models \varphi$.

This is trivially satisfied by the semantics of P_a operator, albeit the semantics are stronger as they are given by a bi-conditional.

The requirement that knowledge is veridical, i.e.

(C.K) If $\Omega, Ms \models K_a\varphi$, then $\Omega, Ms \models \varphi$

likewise make the grade, together with the validity of **T**: $K_a\varphi \to \varphi$. This follows from the reflexivity of $\preceq$, inherited from $\subseteq$. Similarly, $\preceq$ inherits from $\subseteq$ transitivity, yielding

(C.KK*) If $\Omega, Ms \models K_a\varphi$, then $\Omega, Ms \models K_aK_a\varphi$

and thus the validity of **4**: $K_a\varphi \to K_aK_a\varphi$.

As $\preceq$ inherits both reflexivity and transitivity, these validities are no surprise — any modal logic 101 course told you so. But as a matter of fact, noted on page 6, this was also argued by Hintikka: Assuming (C.K) and (C.KK*) is tantamount to assuming the alternativeness relation reflexive and transitive together with the principle

(C.K*) If $\Omega, Ms \models K_a\varphi$ and if $Ms \preceq Nt$, then $\Omega, Nt \models \varphi$.

(C.K*) is already presupposed, in fact, a stronger version of it: It is the left-to-right direction of the semantic clause for K_a.

Two other principles not explicitly endorsed by Hintikka follows under the given semantics, namely the Rule of Necessitation **N**,

$$\frac{\models \varphi}{\models K_a \varphi}$$

conveying that an agent knows all valid formulas and Kripke's axiom schema **K**: $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$ granting that knowledge is closed under implication. **N** preserves truth in all Kripke models and is, together with **K**, what characterizes a *normal* modal logic.

In sum, the K_a and P_a operators satisfy the axioms and inference rules of **S4**. The elements of **S4** not explicitly and formally endorsed by Hintikka are discussed in Section 6.

One principle explicitly rejected by Hintikka is however, as mentioned, **5**: $P_a\varphi \to K_aP_a\varphi$. This principle is not valid under the given semantics, as is well-known from the model theory of modal logic. A counterexample comes right from Hintikka:

> Now it is obviously not excluded by what I now know that I should know more than I now do. But such additional knowledge may very well be incompatible with what now is still possible, as far as I know. (p. 35)

A formal specification of the counterexample may be found in the Appendix.

6 Is Hintikka's Epistemic Logic the Logic of S4?

Casting model systems in the way described above seems to resonate with Hintikka's epistemological ambition of nailing the logic of knowledge by considering the different possibilities for gaining more information. Of course, so do the model set based model systems of *Knowledge and Belief*. The two avenues formally run in parallel and agree on some canonical principles, like **T** and **4**, but they do part company on important points. Here is a selective inventory of such differences based on both the epistemology and mathematics of Hintikka's 1962 treatise. The formal differences are largely by-products of the *partiality* of model sets.

First up, what's the epistemological Hintikkian story with the Rule of Necessitation **N** and axiom schema **K**? Rehearsing Hintikka's initial considerations seem to suggest that both the former and the latter are in play by fiat of indefensibility. Normal modal logics are of course committed to both principles, too, but epistemically they are troublesome in unison: Knowing an infinite number of propositional tautologies and having closure of knowledge under implication amounts to knowledge being closed under logical consequence and here we go *logical omniscience*. To tell a reasonable (and over the ensuing years *long*) story about a quality like logical omniscience one may either mobilize a defense like modelling implicit knowledge in terms of what follows from what one knows or a defense where what is at stake is the characterization of knowledge for idealized agents given conditions of indefensibility. Hintikka may be interpreted in either way [19] at least until the impossible possible worlds are vindicated to counter logical omniscience. Leave impossible possible worlds for a rainy day.

Where **N** and **K** follow in spirit, they follow suit in math. **K** is satisfied in any model system, and **N** preserves truth for all valid φ. Strictly speaking, the principles do hold formally, but not with the same force as in Kripke models. Mathematically, modal logic **S4** is not quite justified as the logic of knowledge from *Knowledge in Belief*, although indeed justified with respect to epistemological intent and ambition. This has to do with the key point of difference.

Model sets are *partial*, and in particular $\{\} = \emptyset$ is a such, given Hintikka's criteria. But if the set of valid formulas is that which is contained in every model set, then **N** may be trivially met, but the reason is empty. As a consequence, $K_a(p \vee \neg p)$ is not a theorem of Hintikka's logic, which can therefore not be **S4**.

The root cause is that partiality causes that correlations between central logical notions like consistency and validity versus Hintikkian defensibility / indefensibility and self-sustainability don't follow run-of-the-mill. Standard links to logic are severed: Some formula φ is consistent iff φ belongs to some model set, so consistency is secured. Now, take validity of a formula φ to mean that it belongs to all model sets corresponding in turn to the self-sustainability of φ. Then it unfortunately doesn't follow that φ is valid if and only if $\neg\varphi$ is inconsistent as one would like to have it. Partiality is yet again the culprit as it is not immediately what the models sets are partial relative to. Information is also lacking as to what the set of atomic propositions check with. Without such information, without more specification on the nature of the relational properties of model sets and without some reasonably straight-forward correspondences between canonical meta-logical properties of the two frameworks it is a bit hard to see immediately how to get a *logic* out of *Knowledge and Belief: An Introduction to the Logic of the Two Notions*.

But history becomes myth, myth becomes legend — also in philosophy — and the robust narrative is that Hintikka is the father of epistemic logic. If emphasis is on logic rather than epistemic, then the narrative is not quite right. In *Knowledge and Belief*, Hintikka is doing epistemology first and foremost. Out of this admirable systematic enterprise, a real logic, in the technical sense, is not among the offsprings. The set of theorems of the system is underdetermined based on what is actually presented in the influential work. But one shouldn't blame Hintikka for this shortcoming. It was never his ambition to this day to lock, stock and barrel the logic, but rather to provide a rigorous treatment of central epistemological notions even if it in the end would mean "Epistemology without Knowledge and without Belief" (2007).

7 Hintikka on the Move

What constitutes Hintikka's epistemic logic may be underdetermined by available evidence. The principles governing his philosophical ideas is another matter. They have — though seemingly unknown — been the objects of intensive study in a recent branch of logic entitled *arbitrary public announcement logic* (APAL) [3, 23, 18, 13, 11].

The connection with APAL stems from taking a slightly different approach to epistemic alternatives than the one above. The different possibilities for an agent to have gained more knowledge is a separate issue from "the different possibilities there are for somebody to gain further information" [16, p. 18] as Hintikka would have it. Accentuation is not on the bulk of knowledge to be had in epistemic alternatives, but on how to informationally get there.

7.1 The Possibilities for Gaining Information

The natural way of measuring amounts of information retained in epistemic states is semantics, by the size of state spaces. The natural way of bringing information to an agent is syntactic, by way of announcing formulas.

Given the introspective nature of **S5** and single agent epistemic states, the informative content of any announcement will be ontic — i.e. purely factual. The informative content of any formula may thus be identified with a Boolean combination φ_B of atomic propositions from Φ. The announcements of these Boolean formulas constitute a superset of "the different possibilities there are for $[a]$ to gain further information".

Not any announcement will do. The literature on dynamic epistemic logic includes prodigious possibilities [5, 20, 6, 4, 8, 12, 21, 22]. The most celebrated so far is *truthful public announcements.* In *public announcement logic* (PAL) [20, 5], the announcement of a formula φ provides all agents with hard, unrevisable information that φ is in fact the case — it is a truthful, public[9] announcement by a fully trusted, and reliable, source.

When an agent claims knowledge and accordingly, on Hintikka's take, confer that no further information would lead to a change of mind, it would seem that circumscription to only true announcements is truly too restrictive. Indeed, the set of announcements *compatible* with the agent's current information should be taken into account. Such leniency far better captures the attitudinal sentiment that the agent's current information is strong enough to claim knowledge. It's exactly the Boolean formulas compatible with the agent's current information that have the potential to further inform the agent. So the set of possibly informative announcements become relative to an epistemic state Ms and consist of the Boolean formulas φ_B the agent considers possible — i.e. that $Ms \models p_a\varphi_B$. Denote this set of possible formulas to be announced Φ_{Ms}.

7.2 Epistemic Alternatives through Announcements

Announcements of informative formulas mileage the inauguration of a model system. Let again $\mathbf{M}_\Phi$ be the set of epistemic states based on the finite set of atomic propositions Φ and let Ms be in $\mathbf{M}_\Phi$. To realize a model system, an alternativeness relation must be defined on $\mathbf{M}_\Phi$, induced somehow by announcements. Denote this relation $\rightsquigarrow$ such that $Ms \rightsquigarrow Nt$ where Nt is an epistemic alternative to Ms. The idea is that Nt is obtainable from Ms along the announcement of a Boolean formula compatible with the agent's information in Ms.

What's to gain — or which epistemic alternatives are then realized when φ_B is announced to the agent? When φ_B is true at Ms, the public announcement update of Ms is surely a reasonable candidate. The effect of updating an epistemic state Ms with φ is a restriction of Ms to those states that satisfy φ — i.e., all $\neg\varphi$ states are without further ado just deleted.

[9] That the announcement is public is of little consequence in a single agent system.

Formally, the restriction of an *unpointed* Kripke model $M = (\llbracket M \rrbracket, \sim, \llbracket \cdot \rrbracket)$ to the set $A \subseteq \llbracket M \rrbracket$ is just the *unpointed* Kripke model $M_{|A} = (\llbracket M_{|A} \rrbracket, \sim_{|A}, \llbracket \cdot \rrbracket_{|A})$ such that

$$\begin{aligned} \llbracket M_{|A} \rrbracket &= \llbracket M \rrbracket \backslash A \\ \sim_{|A} &= \sim \cap \llbracket M_{|A} \rrbracket \times \llbracket M_{|A} \rrbracket \\ \llbracket \cdot \rrbracket_{|A} &= \llbracket \cdot \rrbracket \text{ restricted to the domain } \llbracket M_{|A} \rrbracket \end{aligned}$$

For the restriction of M to $\llbracket \varphi \rrbracket$, the truth set of φ in M, omit brackets and write $M_{|\varphi}$.

The public announcement update of the *pointed* model Ms with φ, for a truthful φ, at s in M is the pointed model $(M_{|\varphi}, s)$. Now $(M_{|\varphi}, s)$ is an epistemic alternative to Ms annexed by the announcement of φ. But is it the only epistemic alternative produced by this update? It seems not. The agent cannot tell the states in M apart; whence, for all states t that survive the announcement, $(M_{|\varphi}, t)$ should be considered epistemic alternatives to Ms. Similar reasoning will have it which epistemic states should be considered epistemic alternatives to Ms as produced by the announcement of a formula φ which is *false* at Ms — namely, the set of epistemic states $(M_{|\varphi}, t)$ for which t survives the update.

Given the set Φ_{Ms} of possible announcements, these considerations yield the following definition of the announcement-based alternativeness relation on $\mathbf{M}_\Phi$:

$$Ms \rightsquigarrow Nt \textit{ iff } \begin{array}{l} 1.\ N = M_{|\varphi} \text{ for some } \varphi \in \Phi_{Ms}, \text{ and} \\ 2.\ t \in \llbracket M_{|\varphi} \rrbracket. \end{array}$$

Given the alternativeness relation, the set of epistemic alternatives to Ms is the set $\{Nt \in \mathbf{M}_\Phi : Ms \rightsquigarrow Nt\}$ and an announcement-based model system is the pair $\Omega^! = (\mathbf{M}_\Phi, \rightsquigarrow)$.

The model system $\Omega^!$ is rooted in epistemological theses from *Knowledge and Belief*. That's good. Better still, just like the model system $\Omega = (\mathbf{M}_\Phi, \preceq)$ defined in Section 5 satisfies the Hintikkian criteria, bang!, the model system $\Omega^!$ does too. In fact

$$\Omega^! = \Omega. \tag{1}$$

The proof is in the Appendix and significant for three distinct reasons:

1. Philosophically, the result goes to show that the two approaches, suggested by Hintikka are, if the modeling is commensurate, equivalent. Hintikka did not suggest that the two approaches were different, but rather used the two points of entry interchangeably. Indeed justified given (1).

2. As to bridge-building, the result demonstrates how Hintikka's epistemological program has strong ties with modern developments in dynamic epistemic logic, i.e. it is fruitious to read his later writings with a dynamic mindset.
3. Technically, pace the developments in dynamic epistemic logic, a sound and complete axiom system for the model systems semantics is pretty much right there for the taking.

7.3 A Logic for Hintikkian Epistemology

In defining the model system $\Omega^! = (\mathbf{M}_\Phi, \rightsquigarrow)$, what has in fact been done is to specify a particular type of *transformation relation* between epistemic states in $\mathbf{M}_\Phi$. In dynamic epistemic logic, such relations are familiar tunes. But the frame obtained — the model system — is typically not viewed from the global perspective adopted here. Rather, the relation is described locally from a given epistemic state. This is accomplished by *dynamic operators*. The operator for the truthful public announcement of φ is $[\varphi]$, and the formula $[\varphi]\psi$ is read 'after the announcement of φ, it is true that ψ'. The semantics are given over epistemic states by

$$Ms \models [\varphi]\psi \text{ iff } Ms \models \varphi \text{ implies } M_{|\varphi}s \models \psi$$

where $M_{|\varphi}$ is the restriction of M to $[\![\varphi]\!]$ as defined above.

The announcement operator $[\varphi]$ captures the effects of the agent receiving the information of φ being the case. Not too surprising then that it is related to the Hintikkian epistemic operators K_a and P_a, interpreted over models systems.

A first observation relates the dual of $[\varphi]$, namely $\langle\varphi\rangle$, to the P_a-operator. The formula $\langle\varphi\rangle\psi$ is satisfied in Ms iff $Ms \models \varphi$ and $M_{|\varphi}s \models \psi$. Thus, by definition, $Ms \rightsquigarrow M_{|\varphi}s$ in $\Omega^!$. Hence, obtained is that

$$Ms \models \langle\varphi\rangle\psi \text{ implies } \Omega^!, Ms \models P_a\psi.$$

Weaker conditions also imply the consequent. For the formula $P_a\psi$ to be true in Ms, it is not required that there exists a restriction of Ms that both contains s and satisfies ψ — only that there be a restriction Nt of Ms such that $\Omega^!, Nt \models \varphi$. Obtained is in turn that

$$Ms \models p_a\langle\varphi\rangle\psi \text{ implies } \Omega^!, Ms \models P_a\psi. \tag{2}$$

The antecedent of (2) may be weakened even further. Though the satisfaction of $P_a\varphi$ requires the existence of a restriction Nt of Ms by *some* announcement, it is not required that this announcement is the particular φ. To state the weaker antecedent, a new operator is required: The arbitrary announcement operator $\Diamond$ introduced in [3]. The formula $\Diamond\psi$ reads 'there exists an announcement such that after it's been made, it is true that ψ', with semantics

$$Ms \models \Diamond\psi \text{ iff } \exists\varphi \in \mathscr{L}_{ep} : Ms \models \langle\varphi\rangle\psi.$$

The sublanguage $\mathscr{L}_{ep}$ contains all Boolean formulas and thus suffices for the announcements under consideration.[10]

Using the $\Diamond$ dynamic modality, the weakened antecedent of (2) is expressible, and it is procured that

$$Ms \models p_a\Diamond\psi \text{ implies } \Omega^!, Ms \models P_a\psi. \tag{3}$$

The antecedent of (3) may not be weakened further. In fact, as this final statement holds true, it's bang on the money:

$$Ms \models p_a\Diamond\psi \text{ if, and only if, } \Omega^!, Ms \models P_a\psi. \tag{4}$$

This statement, established in the Appendix, immediately yields a logic for the Hintikkian epistemic operators K_a and P_a, as they may be defined in terms of the **S5** operators — capturing the information currently blessed with — and the arbitrary announcement operators — capturing the possible ways there are for obtaining new information. The axiom system, based partially on reduction axioms, may be found in the original source on arbitrary public announcements, [3].

8 Back Onboard

In *viva voce* back in Boston some 15 years ago Jaakko relayed to us — while discussing the advent of epistemic logic — the following:

> The epistemology of logic, or the logic of epistemology — it's all the same to me.

[10] The requirement that $\varphi \in \mathscr{L}_{ep}$ — the sublanguage consisting of Boolean and $\{k_a, p_a\}$-epistemic formulas — is to avoid a type of self-reference. See [3] for details.

And to recapitulate Hintikka's word from some of his last writings on epistemic logic and epistemology also some 15 years ago:

> What the concept of knowledge involves in a purely logical perspective is thus a dichotomy of the space of all possible scenarios into those that are compatible with what I know and those that are incompatible with my knowledge. This observation is all we need for most of epistemic logic.

As nonchalant as these statements may sound, they are more pointy on this second round. To Hintikka, epistemology drives logic, and as paradoxical as it may seem, the crown jewels of knowledge and belief are not the true gems even when you are the author of *the* seminal work on epistemic logic exactly entitled *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. As transmitted atop, this is a befitting book title epistemologically but not quite logically. To go even further, also quoting Hintikka yet later, the true gem in "Epistemology without Knowledge and Without Belief" (2007) is *information* of which knowledge, belief and other propositional attitudes, including doubt, are but varieties and special cases. Getting a grip on the dynamic process of acquiring information is more important epistemologically than the static state of having knowledge, belief, certainty or doubt. This driving line of thought is somewhat anticipated back in 1962 if special attention is directed to "the different possibilities there are for somebody to gain further information". Another outstanding contributor to logic, Johan van Benthem, who also has a dedicated volume in this series, has both stressed logic as the science of information processing and coined the vernacular term, *the dynamic turn in logic* [7]. Safe to say that Jaakko Hintikka is back onboard epistemologically and ... (almost) logically too — as the theorem on alternatives and announcements overhead reveals. Not that he ever abandoned ship, but our compass while reading Hintikka was not always pointing in the dynamic direction he may have intended all along.

Bibliography

[1] H. Arló-Costa. Review of V. F. Hendricks, *Mainstream and Formal Epistemology*. *Notre Dame Philosophical Reviews*, 2006.

[2] H. Arló-Costa *(ed.)*, V. F. Hendricks *(ed.)*, J. van Benthem *(ed.)*, H. Boensvang *(asst.ed.)*, and R. K. Rendsvig *(asst.ed.)*. *Readings in Formal Epistemology*. Springer, 2016.

[3] P. Balbiani, A. Baltag, H. van Ditmarsch, A. Herzig, T. Hoshi, and T. de Lima. What Can We Achieve by Arbitrary Announcements?: A Dynamic Take on Fitch's Knowability. In *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge*, TARK'07, pages 42–51. ACM, 2007.

[4] A. Baltag and L. S. Moss. Logics for Epistemic Programs. *Synthese*, 139(2):165–224, 2004.

[5] A. Baltag, L. S. Moss, and S. S. Solecki. The Logic of Public Announcements, Common Knowledge, and Private Suspicions (extended abstract). In *Proceedings of the 7th conference on theoretical aspects of rationality and knowledge (TARK'98)*, pages 43–56. Morgan Kaufmann Publishers, 1998.

[6] A. Baltag and S. Smets. A Qualitative Theory of Dynamic Interactive Belief Revision. In G. Bonanno, W. van der Hoek, and M. Wooldridge, editors, *Logic and the Foundations of Game and Decision Theory (LOFT 7)*, Texts in Logic and Games, Vol. 3, pages 9–58. Amsterdam University Press, 2008.

[7] J. van Benthem. Logic and the Dynamics of Information. *Minds and Machines*, 4(13):503–519, 2003.

[8] J. van Benthem. Dynamic logic for belief revision. *Journal of Applied Non-Classical Logics*, 17(2):129–155, 2007.

[9] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[10] R. Chisholm. The Logic of Knowing. *Journal of Philosophy*, (1963.60):773–795.

[11] M. Cohen. A Dynamic Epistemic Logic with a Knowability Principle. In W. van der Hoek, W. H. Holliday, and W. Wang, editors, *Logic, Rationality, and Interaction*, Lecture Notes in Computer Science, vol. 9394, pages 406–410. Springer, 2015.

[12] H. van Ditmarsch, W. van der Hoek, and Barteld Kooi. *Dynamic Epistemic Logic*. Springer, 2008.

[13] H. van Ditmarsch, S. Knight, and A. Özgün. Arbitrary Announcements on Topological Subset Spaces. In Nils Bulling, editor, *Multi-Agent Systems*, Lecture Notes in Computer Science, vol. 8953, pages 252–266. Springer, 2015.

[14] V. Goranko and M. Otto. Model Theory of Modal Logic. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*. Elsevier, 2008.

[15] V. F. Hendricks. *Mainstream and Formal Epistemology*. Cambridge University Press, 2005.

[16] J. Hintikka. *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. 2nd. ed., College Publications (2005), 1962.

[17] J. Hintikka. Epistemology without Knowledge and without Belief. In *Socratic Epistemology*, pages 11–37. Cambridge University Press, 2007.

[18] L. B. Kuijer. How Arbitrary Are Arbitrary Public Announcements? In Margot Colinet, Sophia Katrenko, and Rasmus K. Rendsvig, editors, *Pristine Perspectives on Logic, Language, and Computation*, Lecture Notes in Computer Science, vol. 8607, pages 109–123. Springer, 2014.

[19] Jago Mark. Hintikka and Cresswell on Logical Omniscience. *Logic and Logical Philosophy*, (15):325–354, 2006.

[20] J. A. Plaza. Logics of public communications. In M. L. Emrich, M. S. Pfeifer, M. Hadzikadic, and Z. W. Ras, editors, *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216, 1989.

[21] R. K. Rendsvig. Pluralistic ignorance in the bystander effect: informational dynamics of unresponsive witnesses in situations calling for intervention. *Synthese*, 191(11):2471–2498, 2014.

[22] R. K. Rendsvig. Model Transformers for Dynamical Systems of Dynamic Epistemic Logic. In W. van der Hoek, W. H. Holliday, and W. Wang, editors, *Logic, Rationality, and Interaction*, Lecture Notes in Computer Science, vol. 9394, pages 316–327. Springer, 2015.

[23] X. Wen, H. Liu, and F. Huang. An Alternative Logic for Knowability. In H. van Ditmarsch, J. Lang, and S. Ju, editors, *Logic, Rationality, and Interaction*, Lecture Notes in Computer Science, vol. 6953, pages 342–355. Springer, 2011.

Appendix: Proofs

Counterexample (p. 14) based on Hintikka quote to show that **5**: $P_a\varphi \to K_aP_a\varphi$ is not valid in the model system $\Omega = (\mathbf{M}_\Phi, \preceq)$.

Assume an epistemic state Ms where the agent does not know p, e.g. with two states, s and t, such that $Ms \models p$ and $Mt \models \neg p$. Then there are two epistemic alternatives to Ms, namely Ks, containing only state s, and Nt containing only t. In the former, a has gained additional knowledge that p. This (new) knowledge is incompatible with what is considered possible now (in Ms), as it is incompatible with Nt and $\neg p$ — to wit, $Ks \models \neg P_a \neg p$. Hence, the epistemic states Ks and Nt are not related by $\preceq$. Hence $\preceq$ is not Euclidean and **5** is not satisfied: $Ms \models P_a \neg p$ due to Nt, but as *not* $Ks \preceq Nt$, it follows that $Ks \not\models P_a \neg p$, so $Ms \not\models K_a P_a \neg p$ and hence $Ms \not\models P_a \neg p \to K_a P_a \neg p$.

Proposition (p. 18):

$$\Omega^! = \Omega.$$

Proof: To establish the claim that $\Omega^! = \Omega$, it suffices to show that $\rightsquigarrow = \preceq$, i.e., that $Ms \rightsquigarrow Nt$ iff $Ms \preceq Nt$.

The **left-to-right** direction follows immediately: $Ms \rightsquigarrow Nt$ implies by definition that $N = M_{|\varphi}$ for some $\varphi \in \Phi_{Ms}$, so $[\![N]\!] \subseteq [\![M]\!]$ and hence by definition $Nt \succeq Ms$.

For the **right-to-left** direction, assume that $Ms \preceq Nt$. It must be established that there exists a $\varphi \in \Phi_{Ms}$ such that 1) $[\![N]\!] = [\![M_{|\varphi}]\!]$ and 2) $t \in [\![M_{|\varphi}]\!]$.

To establish 1), for arbitrary epistemic state (K, m), let

$$\varphi_{Km} := \bigwedge_{p \in \Phi : Km \models p} p \wedge \bigwedge_{q \in \Phi : Km \models \neg q} \neg q.$$

Then φ_{Km} encodes the valuation of m in K. φ_{Km} is well-formed as Φ is finite. Let further

$$\varphi_K := \bigvee_{n \in [\![K]\!]} \varphi_{Km}.$$

φ_K is well-formed as epistemic states are finite. Moreover, $(M, u) \models \varphi_K$ iff $u \in [\![K]\!]$.

A suitable Boolean formula is thus found, namely φ_N: As $(M, u) \models \varphi_N$ iff $u \in [\![N]\!]$, it follows directly that $[\![M_{|\varphi_N}]\!] = [\![N]\!]$.

2) follows immediately: As $Nt \in \mathbf{M}_\Phi$, by definition $t \in [\![N]\!]$ so by 1) $t \in [\![M_{|\varphi}]\!]$. □

Proposition (p. 20):

$$Ms \models p_a \Diamond \psi \text{ if, and only if, } (\Omega^!, Ms) \models P_a \psi.$$

Proof:

Left-to-right: Assume that $Ms \models p_a \Diamond \psi$ for some $Ms \in \mathbf{M}_\Phi$. Then for some $t \in [\![M]\!], Mt \models \Diamond \psi$ and for some $\varphi \in \mathscr{L}_{ep}$, $Mt \models \langle \varphi \rangle \psi$. Hence $M_{|\varphi} t \models \psi$.

As Φ is finite and M is single agent, there exists a Boolean formula φ_B such that $[\![\varphi_B]\!]_M = [\![\varphi]\!]_M$. Hence 1) $\varphi_B \in \Phi_{Mt} = \Phi_{Ms}$ and 2) $M_{|\varphi_B} t \models \psi$. Thus $Ms \rightsquigarrow M_{|\varphi_B} t$, so $\Omega^!, Ms \models P_a\psi$.

Right-to-left: Assume $\Omega^!, Ms \models P_a\psi$. Then there exists a $Nt \in \mathbf{M}_\Phi$ such that $Ms \rightsquigarrow Nt$ and *): $\Omega^!, Nt \models \psi$. But $Ms \rightsquigarrow Nt$ iff 1) $N = M_{|\varphi}$ for some $\varphi \in \Phi_{Ms}$ and 2) $t \in [\![M_{|\varphi}]\!]$. From 2) it follows that $t \in [\![M]\!]$ and from 1) and *) that $Mt \models \langle\varphi\rangle\psi$. Hence $Mt \models \Diamond\psi$, and so $Ms \models p_a\Diamond\psi$. □